IF YOU HAVE ANY QUESTIONS CONCERNING THIS MATERIAL (IN PARTICULAR, SPECIFIC POINTERS TO LITERATURE), PLEASE DON'T HESITATE TO CONTACT ME VIA EMAIL: `trenn@mathematik.uni-kl.de`

4 Equivalence and Quasi-canonical forms

Fact 1: For any invertible matrix $S \in \mathbb{R}^{m\times m}$:

$$(x,u) \text{ solves } E\dot{x} = Ax + Bu \;\Leftrightarrow\; (x,u) \text{ solves } SE\dot{x} = SAx + SBu$$

Fact 2: For coordinate transformation $x = Tz$, $T \in \mathbb{R}^{n\times n}$ invertible:

$$(x,u) \text{ solves } E\dot{x} = Ax + Bu \;\Leftrightarrow\; (z,u) := (T^{-1}x,u) \text{ solves } ET\dot{z} = ATz + Bu$$

Together:

$$(x,u) \text{ solves } E\dot{x} = Ax + Bu \;\Leftrightarrow\; (z,u) := (T^{-1}x,u) \text{ solves } SET\dot{z} = SATz + SBu$$

Definition. (E_1,A_1), (E_2,A_2) are called *equivalent* $:\Leftrightarrow$ $(E_2,A_2) = (SE_1T, SA_1T)$
short:

$$(E_1,A_1) \cong (E_2,A_2)$$

Theorem (Quasi-Kronecker Form)**.** *For any $E, A \in \mathbb{R}^{\ell\times m}$, $\exists$ invertible $S \in \mathbb{R}^{\ell\times\ell}$ and invertible $T \in \mathbb{R}^{n\times n}$:*

$$(E,A) \overset{S,T}{\cong} \left(\begin{bmatrix} E_U & & & \\ & I & & \\ & & N & \\ & & & E_O \end{bmatrix}, \begin{bmatrix} A_U & & & \\ & J & & \\ & & I & \\ & & & A_O \end{bmatrix} \right)$$

where (E_U,A_U) consists of underdetermined blocks on the diagonal, N is nilpotent, and (E_O,A_O) consists of overdetermined diagonal bolcks

Remark: 0×1 underdetermined blocks and 1×0 overdetermined blocks are possible
Example:

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \cong \left(\begin{bmatrix} \overline{} & & \\ & \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} & \\ & & | \end{bmatrix}, \begin{bmatrix} \overline{} & & \\ & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & \\ & & | \end{bmatrix} \right)$$

$$(E,A) \text{ from circuit } \cong \left(\begin{bmatrix} I_{2\times 2} & \\ & 0_{6\times 6} \end{bmatrix}, \begin{bmatrix} \begin{matrix} 0 & 1/C \\ -1/L & -1/RC \end{matrix} & \\ & I_{6\times 6} \end{bmatrix} \right)$$

Corollary. *$E\dot{x} = Ax + f$ has solution x for any sufficiently smooth f and each solution x is uniquely determined by $x(0)$ and f*
$\Leftrightarrow$
$(E,A) \cong \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right)$, *$N$ nilpotent* *Quasi-Weierstrass-Form (QWF)*

(E,A) is then called ***regular*** *(Note: (E,A) regular $\Leftrightarrow$ $\det(sE - A)$ is not the zero polynomial)*

5 Wong sequences

Definition. Let $E, A \in \mathbb{R}^{m\times n}$. The corresponding Wong sequences of the pair (E, A) are:

$$\mathcal{V}_0 := \mathbb{R}^n, \qquad \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i), \qquad i = 0, 1, 2, 3, \ldots$$
$$\mathcal{W}_0 := \{0\}, \qquad \mathcal{W}_{j+1} := E^{-1}A(\mathcal{W}_j), \qquad i = 0, 1, 2, 3, \ldots$$

Note: $M^{-1}\mathcal{S} := \{\ x \mid Mx \in \mathcal{S}\ \}$ and $M\mathcal{S} := \{\ Mx \mid x \in \mathcal{S}\ \}$

Clearly, $\exists i^*, j^* \in \mathbb{N}$

$$\mathcal{V}_0 \supset \mathcal{V}_1 \supset \ldots \supset \mathcal{V}_{i^*} = \mathcal{V}_{i^*+1} = \mathcal{V}_{i^*+2} = \ldots$$
$$\mathcal{W}_0 \subset \mathcal{W}_1 \subset \ldots \subset \mathcal{W}_{j^*} = \mathcal{W}_{j^*+1} = \mathcal{W}_{j^*+2} = \ldots$$

Wong limits:

$$\boxed{\mathcal{V}^* := \bigcap_{i\in\mathbb{N}} \mathcal{V}_i = \mathcal{V}_{i^*}} \qquad \boxed{\mathcal{W}^* = \bigcup_{i\in\mathbb{N}} \mathcal{W}_i = \mathcal{W}_{j^*}}$$

Theorem. *The following statements are equivalent for* square $E, A \in \mathbb{R}^{n\times n}$*:*

(i) (E, A) *is regular*

(ii) $\mathcal{V}^* \oplus \mathcal{W}^* = \mathbb{R}^n$

(iii) $E\mathcal{V}^* \oplus A\mathcal{W}^* = \mathbb{R}^n$

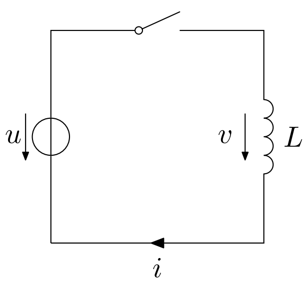
In particular, with $\operatorname{im} V = \mathcal{V}^*$, $\operatorname{im} W = \mathcal{W}^*$

$$(E, A) \textit{ regular} \quad \Rightarrow \quad T := [V, W] \textit{ and } S := [EV, AW]^{-1} \textit{ invertible}$$

and S, T *yield QWF:*

$$(SET, SAT) = \left(\begin{bmatrix} I & \\ & N \end{bmatrix}, \begin{bmatrix} J & \\ & I \end{bmatrix}\right), \ N \textit{ nilpotent}$$

6 Inconsistent initial values: Motivating example



DAE-model: $x = \begin{pmatrix} i \\ v \end{pmatrix}$

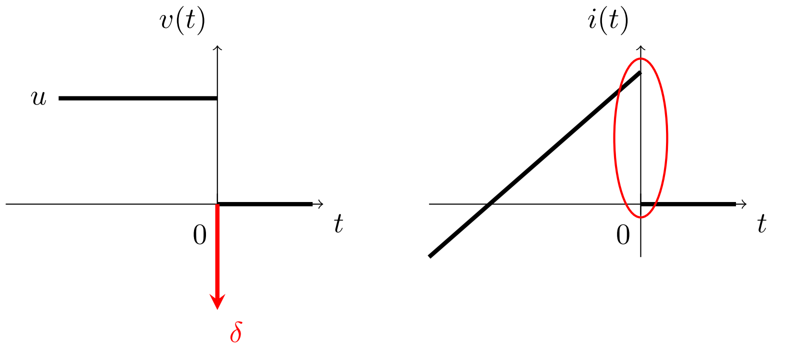
$$\begin{aligned} \text{open switch:} \quad & 0 = i, \\ \text{inductivity law:} \quad & L\tfrac{d}{dt}i = v \end{aligned} \qquad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\dot{x} = x \quad \text{nilpotent DAE}$$

$\Rightarrow$ unique solution $x(t) = 0\ \forall t \in \mathbb{R}$

Now assume switch was opened at $t = 0$, i.e. DAE-model is only valid on $[0, \infty)$.
Different DAE-model for $t < 0$:

$$\begin{aligned} \text{closed switch:} \quad & 0 = v - u, \\ \text{inductivity law:} \quad & L\tfrac{d}{dt}i = v \end{aligned} \qquad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}x + \begin{bmatrix} -1 \\ 0 \end{bmatrix}u$$

Solution (assume constant input u):



Observations:

- $x(0-) = \begin{bmatrix} i(0-) \\ v(0-) \end{bmatrix} \neq 0$ inconsistent for $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \dot{x} = x$
- unique jump from $x(0-)$ to $x(0+)$ (consistent)
- derivative of jump = *Dirac impulse* appears in solution