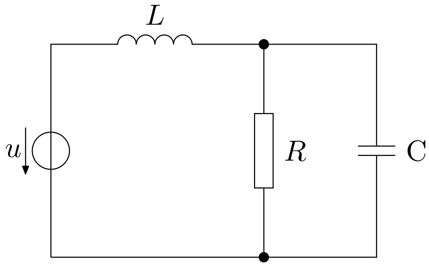
If you have any questions concerning this material (in particular, specific pointers to literature), please don't hesitate to contact me via email: trenn@mathematik.uni-kl.de

1 Motivation: Modeling of electrical circuits



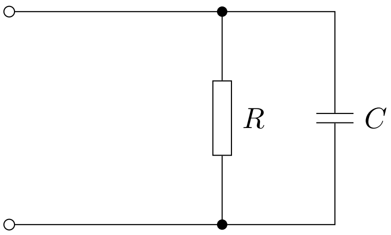
Basic elements:

- Resistors: $v_R(t) = Ri_R(t)$
- Capacitor: $C\frac{d}{dt}v_C(t) = i_C(t)$
- Coil: $L\frac{d}{dt}i_L(t) = v_L(t)$
- Voltage source: $v_S(t) = u(t)$

All components have the same form:

$$\boxed{E\dot{x} = Ax + Bu} \quad E, A \in \mathbb{R}^{\ell\times n},\ B \in \mathbb{R}^{\ell\times m}$$

- Resistor: $x = \begin{pmatrix} v_R \\ i_R \end{pmatrix}$, $E = E_R := [0, 0]$, $A = A_R := [-1, R]$, $B = []$
- Capacitor: $x = \begin{pmatrix} v_C \\ i_C \end{pmatrix}$, $E = E_C := [C, 0]$, $A = A_C := [0, 1]$, $B = []$
- Inductor: $x = \begin{pmatrix} v_C \\ i_C \end{pmatrix}$, $E = E_L := [0, L]$, $A = A_L := [1, 0]$, $B = []$
- Voltage source $x = \begin{pmatrix} v_S \\ i_S \end{pmatrix}$, $E = E_S := [0, 0]$, $A_S := [-1, 0]$, $B = [1]$



Connecting components: Component equations remain unchanged!
+ Kirchhoff's voltage law:

$$v_R + v_C = 0$$

Results again in $E\dot{x} = Ax + Bu$ with $x = (v_R, i_R, v_C, i_C)$ and

$$E = E_{RC} := \begin{bmatrix} \boxed{E_R} & \\ & \boxed{E_C} \\ & \quad\;\; 0 \end{bmatrix}$$

$$A = A_{RC} := \begin{bmatrix} \boxed{A_R} & \\ & \boxed{A_C} \\ 1 \quad\;\; & \text{-1} \quad\;\; \end{bmatrix}$$

Connecting the inductor adds additional constraint:

$$i_L = i_R + i_C, \quad \text{external current} = i_L, \text{ external voltage} = v_L + v_R$$

resulting in $E\dot{x} = Ax + Bu$ with $x = (v_R, i_R, v_C, i_C, v_L, i_L)$ given by

$$E = \begin{bmatrix} \boxed{E_{RC}} & \\ & \boxed{E_L} \\ & \quad\;\; 0 \end{bmatrix}, \quad A = \begin{bmatrix} \boxed{A_{RC}} & \\ & \boxed{A_L} \\ \quad \text{-1} \quad\; \text{-1} & \quad\;\; 1 \end{bmatrix}, \quad B = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Putting all together results in *square* DAE $E\dot{x} = Ax + Bu$ with $x = (v_R, i_R, v_C, i_C, v_L, i_L, v_S, i_S)$ and

$$E = \begin{bmatrix} 0 & 0 & & & & & & \\ & & C & 0 & & & & \\ & & & 0 & & & & \\ & & & & 0 & L & & \\ & & & & & 0 & & \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \\ & & & & & & & 0 \end{bmatrix}, \quad A = \begin{bmatrix} \text{-1} & R & & & & & & \\ & & 0 & 1 & & & & \\ 1 & & \text{-1} & & & & & \\ & & & & 1 & 0 & & \\ & \text{-1} & & \text{-1} & & 1 & & \\ & & & & & & \text{-1} & 0 \\ & & & & & \text{-1} & & 1 \\ \text{-1} & & & & \text{-1} & & 1 & \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

2 DAEs: Differences to ODEs

Recall properties of ODEs $\qquad \dot{x} = Ax + f$

- Initial values: arbitrary
- Solution uniquely determined by f and $x(0)$
- No inhomogeneity constraints

DAE example:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

$$\begin{aligned} \dot{x}_2 &= x_1 + f_1 \longrightarrow x_1 = -f_1 - \dot{f}_2 \\ 0 &= x_2 + f_2 \longrightarrow x_2 = -f_2 \nearrow \\ 0 &= f_3 \end{aligned}$$

no restriction on x_3

Observations:

- For fixed inhomogeneity, initial values cannot be chosen arbitrarily ($x_1(0) = -f_1(0) - \dot{f}_2(0)$, $x_2(0) = f_2(0)$)
- For fixed inhomogeneity, solution not uniquely determined by initial value (x_3 free)
- Inhomogeneity constraints
  - structural restrictions ($f_3 = 0$)
  - differentiability restrictions ($\dot{f}_2$ must be well defined)

3 Special DAE-cases

a) ODEs ✓

b) nilpotent DAEs:

$$\begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \dot{x} = x + f$$

$$\begin{aligned} \Leftrightarrow \quad 0 &= x_1 + f_1 & \longrightarrow \quad & x_1 = -f_1 \\ \dot{x}_1 &= x_2 + f_2 & \longrightarrow \quad & x_2 = -f_2 - \dot{f}_1 \\ \dot{x}_2 &= x_3 + f_3 & \longrightarrow \quad & x_3 = -f_3 - \dot{f}_2 - \ddot{f}_1 \\ &\vdots & \vdots \quad & \vdots \\ \dot{x}_{n-1} &= x_n + f_n & \longrightarrow \quad & x_n = -\sum_{i=1}^{n} f_i^{(n-i)} \end{aligned}$$

In general:

$$\begin{aligned} & N\dot{x} = x + f \quad \text{with } N \text{ nilpotent, i.e. } N^n = 0 \\ & \overset{N\frac{d}{dt}}{\Rightarrow} N^2\ddot{x} = N\dot{x} + N\dot{f} = x + f + N\dot{f} \\ & \overset{N\frac{d}{dt}}{\Rightarrow} N^3\dddot{x} = N^2\ddot{x} + N^2\ddot{f} = x + f + N\dot{f} + N^2\ddot{f} \\ & \vdots \\ & \overset{N\frac{d}{dt}}{\Rightarrow} \underbrace{N^n x^{(n)}}_{=0} = x + \sum_{i=0}^{n-1} N^i f^{(i)} \\ & \Rightarrow x = -\sum_{i=0}^{n-1} N^i f^{(i)} \end{aligned}$$

is unique solution of $N\dot{x} = x + f$

- Initial values: *fixed* by inhomogeneity
- Solution uniquely determined by f
- Inhomogeneity constraints:
  - no structural constraints
  - differentiability constraints: $\sum_{i=0}^{n-1} N^i f^{(i)}$ needs to be well defined

c) underdetermined DAEs

$$\underbrace{\begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}}_{n}{}_{n-1} \dot{x} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} x + f$$

$$\Leftrightarrow \begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_{n-1} \end{pmatrix} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_n \end{pmatrix} + f$$

$\Leftrightarrow$ ODE with additional "input" x_n

- Initial values: arbitrary
- Solution *not uniquely* determined by $x(0)$ and f
- Inhomogeneity constraints: none

d) overdetermined DAEs

$$\underbrace{\begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & \ddots & 0 \\ & & & 1 \end{bmatrix}}_{n}{}_{n+1} \dot{x} = \begin{bmatrix} 1 & & & \\ 0 & \ddots & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} x + f$$

$$\Leftrightarrow \underbrace{\begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}}_{N} \dot{x} = x + \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \ \wedge\ \dot{x}_n = f_{n+1}$$

$$\Leftrightarrow x = -\sum_{i=0}^{n-1} N^i f^{(i)} \ \wedge\ \underbrace{\dot{x}_n = -\sum_{i=1}^{n} f_i^{(n-i+1)} \stackrel{!}{=} f_{n+1}}_{\Leftrightarrow \sum_{i=1}^{n+1} f_i^{(n+1-i)} = 0}$$

- Initial valus: fixed by inhomogeneity
- Solution uniquely determined by f
- Inhomogeneity constraints
  - structural constraint: $\sum_{i=1}^{n+1} f_i^{(n+1-i)} = 0$
  - differentiability constraint: $f_i^{(n+1-i)}$ needs to be well defined

We will see: There are *no other cases*!