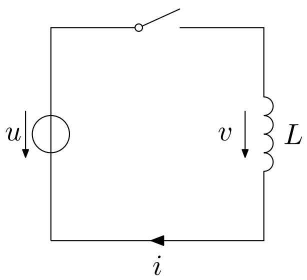
If you have any questions concerning this material (in particular, specific pointers to literature), please don't hesitate to contact me via email: trenn@mathematik.uni-kl.de

10 Switched DAEs

10.1 Motivation and solutions

Recall example from Lecture 3:



Switch → Different DAE models (=modes) depending on (time-varying) position of switch

Switching signal $\sigma : \mathbb{R} \to \{1, \ldots, N\}$ picks mode number $\sigma(t)$ at each time $t \in \mathbb{R}$:

$$\begin{aligned} E_{\sigma(t)}\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) &= C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t) \end{aligned}$$

or short

$$\boxed{\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned}} \tag{swDAE}$$

Each mode might have different consistency spaces
⇒ inconsistent initial values at each switch
⇒ distributional solutions

In (swDAE) multiplication of piecewise-constant function with distribution appears.

Lemma 1 (Multiplication of distributions)**.**

- *Let $\alpha \in \mathcal{C}^\infty$ and $D \in \mathbb{D}$, then $\alpha \cdot D \in \mathbb{D}$ where*

$$(\alpha \cdot D)(\varphi) := D(\alpha \cdot \varphi)$$

- *Let $\alpha = \sum_{i\in\mathbb{Z}} \alpha_{i[t_i,t_{i+1})} \in \mathcal{C}^\infty_{\mathrm{pw}}$ and $D \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$, then $\alpha \cdot D \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ where*

$$\alpha \cdot D := \sum_{i\in\mathbb{Z}} \alpha_i \cdot D_{[t_i,t_{i+1})},$$

in particular, $\mathbb{1}_{[0,\infty)} \cdot \delta = \delta$.

Remarks 1.

a) It is *not possible* to define *commutative* multiplication $F * G$ neither for general $F,G \in \mathbb{D}$ nor for $F,G \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ (→ Exercise)

b) A noncommutative mutliplication $F \cdot G$ for $F,G \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ can be defined, in particular,

$$\delta \cdot \delta = 0$$

(because $\delta \cdot \delta = \mathbb{1}'_{[0,\infty)} \cdot \delta = (\mathbb{1}_{[0,\infty)} \cdot \delta)' - \mathbb{1}_{[0,\infty)} \cdot \delta' = \delta' - \delta' = 0$)

Corollary 1 (from Lecture 3)**.** *Let*

$$\Sigma_0 := \left\{ \sigma : \mathbb{R} \to \{1,\ldots,N\} \;\middle|\; \sigma \text{ is piecewise constant and } \sigma\big|_{(-\infty,0)} \text{ is constant} \right\}.$$

Consider (swDAE) with regular (E_p,A_p) $\forall p \in \{1,\ldots,N\}$. *Then for all* $u \in \mathbb{D}^m_{\mathrm{pw}\mathcal{C}^\infty}$ *and all* $\sigma \in \Sigma_0$ *exists solution* $x \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ *of (swDAE) and* $x(0-)$ *uniquely determines* x.

10.2 Impulse-freeness

Question: When are all solutions of homogenous (swDAE) $E_\sigma \dot{x} = A_\sigma x$ *impulse free*, i.e. $x[t] := x_{[t,t]} = 0$ $\forall t \in \mathbb{R}$?
(jumps are OK)

Lemma 2 (Sufficient conditions)**.**

- (E_p,A_p) all *have index one (i.e.* $N_p = 0$ *in QWF)*
  $\Rightarrow$ *(swDAE) impulse free*
- *all consistency spaces of* (E_p,A_p) *coincide (i.e. Wong limits* $\mathcal{V}_p^*$ *are identical)*
  $\Rightarrow$ *(swDAE) impulse free*

Proof:

- Index-1-case: Consider nilpotent DAE-ITP:

  $$\begin{aligned} (N\dot{w})_{[0,\infty)} &= w_{[0,\infty)} \\ \Rightarrow 0 &= w_{[0,\infty)} \\ \Rightarrow w[0] &= 0 \end{aligned}$$

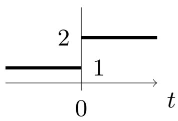
  Hence an inconsistent initial value does not induce Dirac-impulse

- Same consistency space for all modes
  $\Rightarrow$ no inconsistent initial values at switch
  $\Rightarrow$ no Dirac-impulse

Theorem 1. *The switched DAE* $E_\sigma \dot{x} = A_\sigma x$ *is impulse free* $\forall \sigma \in \Sigma_0$

$$\Leftrightarrow \quad E_q(I - \Pi_q)\Pi_p = 0 \quad \forall p,q \in \{1,\ldots,N\}$$

where $\Pi_p := \Pi_{(E_p,A_p)}$, $p \in \{1,\ldots,N\}$ *is the consistency projector.*

Proof: It suffices to consider $\sigma =$ 

i.e. (swDAE) reads as

$$(E_1\dot{x})_{(-\infty,0)} = (A_1 x)_{(-\infty,0)} (E_2\dot{x})_{[0,\infty)} \quad = (A_2 x)_{[0,\infty)} \tag{$*$}$$

Choose S_2,T_2 invertible such that $(S_2E_2T_2, S_2A_2T_2)$ is in QWF, then $(*)$ is equivalent to

$$(\widetilde{E}_1\dot{z})_{(-\infty,0)} = (\widetilde{A}_1 z)_{(-\infty,0)} \left(\begin{bmatrix} I & \\ & N \end{bmatrix} \dot{z} \right)_{[0,\infty)} = \left(\begin{bmatrix} J & \\ & I \end{bmatrix} z \right)_{[0,\infty)}$$

where $z = T_2^{-1}x$ and $(\widetilde{E}_1,\widetilde{A}_1) = (S_2E_1T_2, S_2A_1T_2)$. Note that $z(0-) = T_2^{-1}x(0-) \in T_2^{-1}\mathrm{im}\Pi_1$. Let $z = \begin{pmatrix} v \\ w \end{pmatrix}$ then $(*)$ is impulse free

$\Leftrightarrow$ ITP for $N\dot{w} = w$ is impulse free for all $\boxed{w(0-) \in [0, I]T_2^{-1}\operatorname{im}\Pi_1}$.

Since $w[0] = -\sum_{i=0}^{n-2} N^{i+1}w(0-)\delta^{(i)}$ we have

$$\begin{aligned}
(*) \text{ is impulse free} \quad &\Leftrightarrow \quad N^{i+1}[0,I]T_2^{-1}\operatorname{im}\Pi_1 \ \forall i \in \{0,1,\dots,n-2\} \\
&\Leftrightarrow \quad N[0,I]T_2^{-1}\Pi_1 = 0 \\
&\Leftrightarrow \quad \begin{bmatrix} I & \\ & N \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} T_2^{-1}\Pi_1 = 0 \\
&\Leftrightarrow \quad S_2^{-1} \begin{bmatrix} I & \\ & N \end{bmatrix} T_2^{-1}T_2 \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} T_2^{-1}\Pi = 0 \\
&\Leftrightarrow \quad E_2(I - \Pi_2)\Pi_1 = 0
\end{aligned}$$

Remarks 2.

a) Index 1 $\Leftrightarrow E_p(I - \Pi_p) = 0 \ \forall p$

b) Consistency spaces equal $\Leftrightarrow (I - \Pi_q)\Pi_p = 0 \ \forall p,q$

10.3 Stability

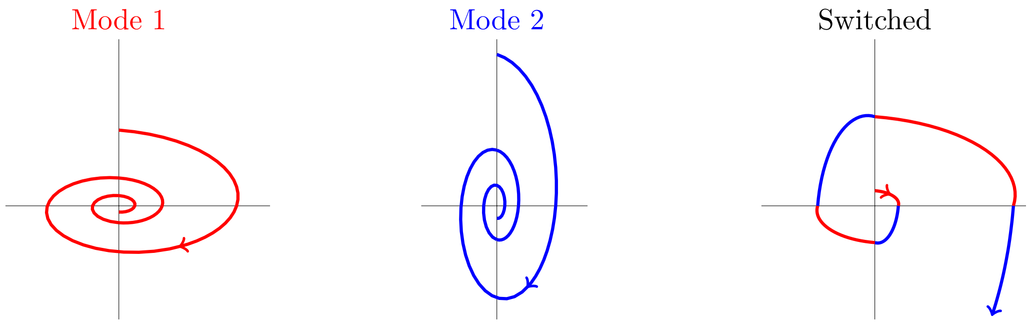
Definition 1. $E_\sigma\dot{x} = A_\sigma x$ is called (asymptotically) stable (for given σ) :$\Leftrightarrow$

1) all solutions are *impulse free*

2) $x(t\pm) \to 0$ as $t \to \infty$

Question: When is $E_\sigma\dot{x} = A_\sigma x$ stable $\forall\sigma$?

Attention: Stability of each mode $E_p\dot{x} = A_p x$ is necessary but *not sufficient*, ODE-example:

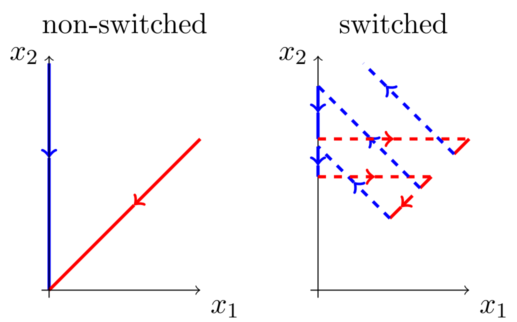


For switched DAEs jumps play also important role:

Examples 1.

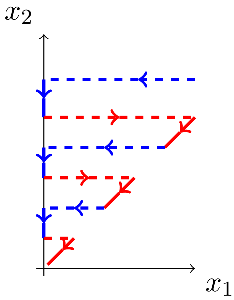
a) $E_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $A_1 = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$

$E_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$



→ jumps destabilize

b) (E_1,A_1) as above, $E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
non-switched behavior exactly the same as above, but switched behavior now stable:



Proposition 1. *$E\dot{x} = Ax$ asymptotically stable for regular (E,A)*
⇔ generalized Lyapunov equation

$$A^\top PE + E^\top PA = -Q \qquad (*)$$

has solution (P,Q) with $P = P^\top > 0$ (positiv definite) and $Q = Q^\top$ positiv definite on consistency space.
In particular, $E\dot{x} = Ax$ asymptotically stable
⇔ ∃ Lyapunov Function

$$V(x) = (Ex)^\top PEx$$

where P is solution of $(*)$ for some Q.

Note that

$$\begin{aligned} \tfrac{d}{dt}V(x(t)) &= (Ex(t))^\top PE\dot{x}(t) + (E\dot{x}(t))^\top PEx(t) \\ &= x(t)^\top E^\top PAx(t) + x(t)A^\top PEx(t) \\ &= -x(t)Qx(t) < 0 \end{aligned}$$

Theorem 2. *$E_\sigma\dot{x} = A_\sigma x$ is asymptotically stable $\forall\sigma$ if*

1) *$E_q(I - \Pi_q)\Pi_p = 0$ $\forall p,q$ (impulse freeness)*

2) *∃ Lyapunov Function $V_p(x) = (E_px)^\top P_pE_px$ $\forall p$ (each mode asymptotically stable)*

3) *$\forall p,q \in \{1,\ldots,N\} \forall x \in \operatorname{im}\Pi_p$:*

$$V_q(\Pi_q x) \leq V_p(x) \qquad (**)$$

Note that for all $x \in \Pi_p \cap \Pi_q$:

$$V_q(x) = V_q(\Pi_q x) \leq V_p(x) = V_p(\Pi_p x) \leq V_q(x)$$

⇒ $V_q(x) = V_p(x)$ on intersection of consistency space
⇒ $(**)$ generalizes the well-known "common Lyapunov function" condition of switched ODEs.

Remark 3. Result also holds for nonlinear switched DAEs:

$$E_\sigma(x)\dot{x} = f_\sigma(x).$$