Distributional averaging of switched DAEs with two modes

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***Abstract*— The averaging technique is a powerful tool for the analysis and control of switched systems. Recently, classical averaging results were generalized to the class of switched differential algebraic equations (switched DAEs). These results did not consider the possible Dirac impulses in the solutions of switched DAEs and it was believed that the presence of Dirac impulses does not prevent convergence towards an average model and can therefore be neglected. It turns out that the first claim (convergence) is indeed true, but nevertheless the Dirac impulses cannot be neglected, they play an important role for the resulting limit. This note first shows with a simple example how the presence of Dirac impulses effects the convergence towards an averaged model and then a formal proof of convergence in the distributional sense for switched DAEs with two modes is given.**

I. Introduction

This note considers averaging of switched differential-algebraic equations (DAEs) with two modes of the form

$$E_\sigma \dot{x} = A_\sigma x, \tag{1}$$

where $\sigma : \mathbb{R} \to \{1, 2\}$ is the switching signal and $E_1, A_1, E_2, A_2 \in \mathbb{R}^{n\times n}$. Switched DAEs are a canonical modeling framework to study dynamical systems with algebraic constraints (e.g. the Kirchhoff laws in electrical circuits) which are subject to sudden structural changes (e.g. faults or switches in electrical circuits), see [15].

The idea of averaging is based on the observation that the trajectories of a switched system approach the trajectories of an averaged non-switched system when the switching frequency increases. An application of averaging could be stabilization via fast switching, because it is possible that each mode is unstable, but the average system is stable. In general, the analysis of the switched system simplifies significantly when it can be sufficiently well approximated by an averaged system.

For switched (linear) ordinary differential equations (ODEs), i.e. where $E_\sigma = I$ in (1), it is well known (see e.g. [2] or [3]) that convergence towards an average model is always guaranteed. However, due to the presence of jumps in the solutions of switched DAEs (1) convergence towards an average system does not always takes place and additional assumptions have to be made [5], [4], [9], [7], [8]. In addition to jumps, the solutions of (1) can also contain Dirac impulse (the observation that inconsistent initial values can induce Dirac impulses was already made in [17], for a recent discussion on the effect of inconsistent initial values see [16, Sec. 3]). The available averaging results for switched DAEs do not allow for Dirac impulses in the solutions and in [5, Rem. 1] the hope was articulated that the effect of the Dirac impulses for fast switching can be neglected, because the Dirac impulses are induced by the jumps in the solutions and the magnitude of the jumps converges to zero for an increasing switching frequency. *Unfortunately, this is not true.* Nevertheless, it is possible to show convergence in a distributional sense of the trajectories of the switched DAE towards trajectories induced by an non-switched average system, which is the major novel contribution of this note.

The structure of this note is as follows: First some mathematical preliminaries are recalled. Afterwards, a simple example is discussed which shows that Dirac impulse cannot be neglected. In Section IV the main result concerning the distributional convergence of the trajectories of the switched DAE for increasing switching frequency is shown.

II. Mathematical preliminaries

A. Distribution theory

The space of distributions is

$$\mathbb{D} := \{\ D : \mathcal{C}_0^\infty \to \mathbb{R} \ \mid D \text{ is linear and continuous }\},$$

where $\mathcal{C}_0^\infty$ is the space of test functions [11], i.e. the space of smooth functions with compact support, where the support of $\varphi : \mathbb{R} \to \mathbb{R}$ is defined as

$$\operatorname{supp} \varphi := \overline{\{\ t \in \mathbb{R} \ \mid \varphi(t) \neq 0\ \}}.$$

Note that continuity of $D : \mathcal{C}_0^\infty \to \mathbb{R}$ is only well defined with respect to a certain locally convex topology on $\mathcal{C}_0^\infty$, see e.g. [10, Def. 6.3 & Thm. 6.4]. Distributions are also called generalized functions because any locally integrable function $f : \mathbb{R} \to \mathbb{R}$ induces a distribution as follows:

$$f_\mathbb{D} : \mathcal{C}_0^\infty \to \mathbb{R}, \varphi \mapsto \int_\mathbb{R} \varphi f.$$

The Dirac impulse (a.k.a. Dirac Delta or Delta "function") at $t \in \mathbb{R}$ is not induced by any function and is formally defined as

$$\delta_t : \mathcal{C}_0^\infty \to \mathbb{R}, \quad \varphi \mapsto \delta_t(\varphi) = \varphi(t).$$

Distributions are always differentiable with derivative

$$D'(\varphi) := -D(\varphi').$$

This derivative generalizes the standard differentiation of differentiable functions, because integration by parts yields $(f')_\mathbb{D} = (f_\mathbb{D})'$ for any differentiable function $f : \mathbb{R} \to \mathbb{R}$. The Dirac impulse δ_0 is the distributional derivative of the Heaviside step function, this motivated the common "definition" of the Dirac impulse as $\delta_0(t) = 0$ for $t \neq 0$ and

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$\delta_0(0) = \infty$ with $\int_{\mathbb{R}} \delta_0 = 1$; although not formally correct, this "definition" helps to visualize the Dirac impulse as an infinite peak.

It is well known (see e.g. [10, Thm. 6.17]) that a sequence $(D_n)_{n\in\mathbb{N}}$ of distributions converges if, and only if, for each test function $\varphi \in \mathcal{C}_0^\infty$ the sequence $(D_n(\varphi))_{n\in\mathbb{N}}$ of real numbers converges; in particular, the limit given by $D(\varphi) := \lim_{n\to\infty} D_n(\varphi)$ is again a distribution (i.e. linear and continuous). In this note convergence on an interval $\mathfrak{I} \subseteq \mathbb{R}$ will be of interest, which means that $D(\varphi) = \lim_{n\to\infty} D_n(\varphi)$ is only considered for test functions whose support is contained in $\mathfrak{I}$; the notation for this "restricted" convergence is

$$D_n \underset{\mathbb{D}_{\mathfrak{I}}}{\longrightarrow} D.$$

The whole space of distribution is not suitable as a solution space for switched DAEs because the product of a piecewise-constant function with a general distribution is not well defined. To overcome this problem, one can introduce the smaller space of piecewise-smooth distributions [14]

$$\mathbb{D}_{\text{pw}\mathcal{C}^\infty} := \left\{ D = f_{\mathbb{D}} + \sum_{t\in T} D_t \;\middle|\; \begin{array}{l} f \in \mathcal{C}^\infty_{\text{pw}}, T \subseteq \mathbb{R} \text{ is discrete} \\ \forall t \in T: D_t \in \operatorname{span}\{\delta_t, \delta_t', \delta_t'', \ldots\} \end{array} \right\},$$

where $\mathcal{C}^\infty_{\text{pw}}$ is the space of piecewise-smooth functions, i.e. functions of the form $f = \sum_{i\in\mathbb{Z}} (f_i)_{[s_i, s_{i+1})}$, where f_i, $i \in \mathbb{Z}$ are smooth and $\{ s_i \mid i \in \mathbb{Z} \}$ is ordered and locally finite. In other words, a piecewise-smooth distribution D consists of a function part $f_{\mathbb{D}}$ and a purely impulsive part $D[\cdot] := \sum_{t\in T} D_t$ which contains Dirac-impulses (and its derivates) at isolated points in time, see Figure 1

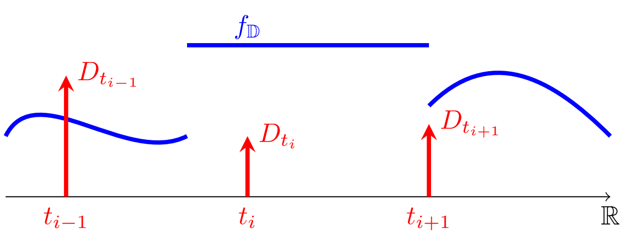


Fig. 1. Illustration of a piecewise-smooth distribution $D = f_{\mathbb{D}} + \sum_{t\in T} D_t$, Dirac impulses (and there derivatives) are shown as arrows.

For a piecewise-smooth distribution D left- and right-evaluation is well defined via $D(t^\pm) := f(t^\pm) := \lim_{\varepsilon\searrow 0} f(t\pm\varepsilon)$ as well as the impulsive evaluation $D[t] := D_t$ if $t \in T$ and zero otherwise.

B. Solution theory of switched DAEs

A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is called regular, if the polynomial $\det(sE - A)$ is not the zero polynomial; and the switched DAE (1) is called regular if both matrix pairs (E_1, A_1) and (E_2, A_2) are regular. In [15] it is shown that the regular switched DAE (considered on $[0, \infty)$) has for any initial condition $x(0^-) = x^0 \in \mathbb{R}^n$ and any switching signal (without finite accumulation of switching times) a unique solution $x \in \mathbb{D}^n_{\text{pw}\mathcal{C}^\infty}$. In particular, the jumps and Dirac impulses induced by the switches are uniquely determined. For explicit solution formulas the quasi-Weierstrass form (QWF) of a regular matrix pair (E, A) is helpful:

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{2}$$

where S, T are invertible matrices and N is nilpotent. The quasi-Weierstrass form can easily be calculated via the Wong sequences [1]. Based on the QWF the consistency projector of (E, A) is [14]

$$\Pi := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1};$$

the flow and impulse matrix of (E, A) are [12]

$$A^{\text{diff}} := T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1},$$
$$E^{\text{imp}} := T \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} T^{-1}.$$

Note that although the QWF transformation matrices S and T are not unique, the resulting consistency projector, flow and impulse matrices are unique.

In this note the following periodic switching signal is considered:

$$\sigma_p(t) = \begin{cases} 1, & t \in [kp, kp + d_1 p),\ k \in \mathbb{N} \\ 2, & t \in [kp + d_1 p, (k+1)p),\ k \in \mathbb{N} \end{cases} \tag{3}$$

with period $p > 0$ and duty cycle $0 < d_1 < 1$ of the first mode ($d_2 := 1 - d_2$ will denote the duty cycle of the second mode in the following). Let $\Pi_i, A_i^{\text{diff}}, E_i^{\text{imp}}$, $i = 1, 2$ denote the consistency projector, flow and impulse matrix of the regular matrix pair (E_i, A_i). Then the impulse-free part $x_p^f := x_p - x_p[\cdot]$ of the distributional solution x_p of (1) satisfies

$$x_p^f(t_k^-) = \left(\mathrm{e}^{A_2^{\text{diff}} d_2 p} \Pi_2 \mathrm{e}^{A_1^{\text{diff}} d_1 p} \Pi_1 \right)^k x^0, \tag{4}$$

where $t_k := kp$, $k \in \mathbb{N}$, and for $\tau \in [0, p)$

$$x_p^f((t_k + \tau)^+) = M_p(\tau) x_p^f(t_k^-),$$

where

$$M_p(\tau) = \begin{cases} \mathrm{e}^{A_1^{\text{diff}} \tau} \Pi_1, & \tau \in [0, d_1 p), \\ \mathrm{e}^{A_2^{\text{diff}} (\tau - d_1 p)} \Pi_2 \mathrm{e}^{A_1^{\text{diff}} d_1 p} \Pi_1, & \tau \in [d_1 p, p). \end{cases}$$

The impulses at the switching times t_k and $s_k := t_k + d_1 p$ are given by ([12, Cor. 5], c.f. [13, Rem. 6]):

$$x_p[t_k] = -\sum_{i=0}^{n-2} (E_1^{\text{imp}})^{i+1} x_p^f(t_k^-) \delta_{t_k}^{(i)},$$

$$x_p[s_k] = -\sum_{i=0}^{n-2} (E_2^{\text{imp}})^{i+1} x_p^f(s_k^-) \delta_{s_k}^{(i)}.$$

III. An illustrative example

Consider the switched DAE (1) with modes

$$(E_1, A_1) = \left(\begin{bmatrix} 1 & & \\ & 0 & 0 \\ & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & & \\ & 1 & 0 \\ & 0 & 1 \end{bmatrix} \right),$$

$$(E_2, A_2) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

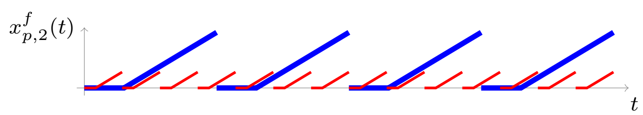
and switching signal (3). Note that both modes are already in QWF, i.e. $S = T = I$ in (2). Since $\dot{x}_{p,1} = 0$ in both modes it is clear that the first component $x_{p,1}$ of the solution x_p satisfies $x^f_{p,1}(t) = x^0_1 \in \mathbb{R}$ and $x_{p,1}[\cdot] = 0$. Furthermore, on the open intervals between the switching times it is easily seen that $x^f_{p,3}(t) = 0$ in both modes and $x^f_{p,2}(t) = 0$ in mode one and

$$x^f_{p,2}(s_k + \tau) = x^0_1 \tau, \quad \tau \in (0, d_2 p)$$

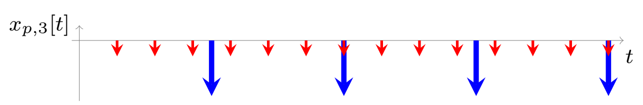
in mode two, see Figure 2a for $d_1 = 0.3$. Clearly, with an increasing switching frequency the jump magnitude converges to zero. Since $\dot{x}_{p,2} = x_{p,3}$ in mode one, it follows that the jumps in $x_{p,2}$ induce Dirac impulses in $x_{p,3}$, see Figure 2b. In particular

$$x_{p,3} = x_{p,3}[\cdot] = -\sum_{k=1}^{\infty} d_2 p x^0_1 \delta_{kp}.$$

Clearly, the magnitude of the Dirac impulses converge to



(a) Second component $x_{p,2} = (x^f_{p,2})_{\mathbb{D}}$; large switching period p (blue, thick), small p (red, thin).



(b) Impulsive part (Dirac impulses are visualized as arrows whose length reflect its magnitude) of third component $x_{p,3} = x_{p,3}[\cdot]$; large switching period p (blue, thick), small p (red, thin).

Fig. 2. Second and third component of the solution of the switched DAE from Section III.

zero proportionally with the decreasing period p, but at the same time the number of Dirac impulses in any interval increases linearly with the increasing frequency $1/p$. The question is now: Does the impulsive part $x_3[\cdot]$ converges to zero in the *distributional* sense (as hoped in [5])?

For any $\varphi \in \mathcal{C}^\infty_0$ with $\operatorname{supp} \varphi \subseteq [0, T]$, $T \in \mathbb{R}$, it holds that

$$x_{p,3}[\cdot](\varphi) = -\sum_{k=1}^{\lfloor T/p \rfloor} d_2 p x^0_1 \delta_{kp}(\varphi) = -d_2 x^0_1 \sum_{k=1}^{\lfloor T/p \rfloor} p\varphi(kp).$$

But the latter is just the (right) Riemann sum of φ on the interval $[0, T]$ and convergence is guaranteed, since φ is smooth (and therefore in particular Riemann integrable), hence (with $\mathbb{1} : \mathbb{R} \to \mathbb{R}, t \mapsto 1$)

$$x_{p,3}[\cdot](\varphi) \overset{p \to 0}{\longrightarrow} -d_2 x^0_1 \int_0^T \varphi = (-d_2 x^0_1) \mathbb{1}_{\mathbb{D}}(\varphi),$$

i.e. *the impulsive part converges to the nonzero constant function* $-d_2 x^0_1$. In particular, the Dirac impulses cannot be neglected for an increasing switching frequency!

This example showed that although the distributional limit does not contain Dirac impulses, the limit of the impulse part does not vanish. This seems counterintuitive and its relevance for real world scenarios is not clear. Of course there are no Dirac impulses in the real world, but only very high peaks. For example, a non-ideal Dirac impulse could be given by, for $\varepsilon > 0$,

$$\delta^\varepsilon_t(\tau) = \begin{cases} 0, & \tau \notin [t - \varepsilon, t + \varepsilon], \\ \frac{1}{\varepsilon} - \frac{|\tau - t|}{\varepsilon^2}, & \tau \in [t - \varepsilon, t + \varepsilon] \end{cases}$$

The corresponding "realistic" third component of the solution of the example is then given by

$$x_{p,3}[\cdot]^\varepsilon = \sum_{k=1}^{\lfloor T/p \rfloor} d_2 p x^0_1 \delta^\varepsilon_{kp} \tag{5}$$

and its behavior for large, medium and small p is shown in Figure 3. Clearly the individual non-ideal Dirac impulses converge to zero as p gets smaller, however due to the overlapping of the supports the sum (5) does not converge to zero.

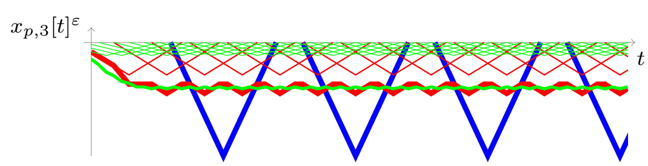


Fig. 3. Non-ideal impulse solution $x_{p,3}[\cdot]^\varepsilon$ for large switching period p (thick, blue), medium p (thick, red) and small p (green); thin red and green lines are the individual non-ideal Dirac impulses $\delta^\varepsilon_{t_k}$.

IV. Main result

Theorem 1: Consider the regular switched DAE (1) with initial condition $x(0^-) = x^0 \in \mathbb{R}^n$ and periodic switching signal $\sigma = \sigma_p$ given by (3) with period $p > 0$. Let A^{diff}_i, E^{imp}_i, Π_i, $i = 1, 2$ be the flow matrix, impulse matrix and consistency projector of the corresponding matrix pair (E_i, A_i). Assume that the consistency projectors commute, i.e.

$$\Pi_\cap := \Pi_1 \Pi_2 = \Pi_2 \Pi_1.$$

Then the solution $x_p = (x^f_p)_{\mathbb{D}} + x_p[\cdot] \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ of the switched DAE converges to the solution $x_{\text{av}} : \mathbb{R} \to \mathbb{R}$ of the average ODE

$$\dot{x}_{\text{av}} = A^{\text{diff}}_{\text{av}} x, \quad x_{\text{av}}(0) = \Pi_\cap x^0,$$

where $A^{\text{diff}}_{\text{av}} := \Pi_\cap (d_1 A^{\text{diff}}_1 + d_2 A^{\text{diff}}_2) \Pi_\cap$, in the following sense:

(i) For any compact $\mathfrak{I} \subseteq (0,\infty)$:
$$\|(x_p^f - x_{\text{av}})_{\mathfrak{I}}\|_\infty = O(p).$$

(ii) For any compact $\mathfrak{I} \subseteq (0,\infty)$:
$$x_p[\cdot] \xrightarrow[\mathbb{D}_{\mathfrak{I}}]{} -E_{\text{av}}^{\text{imp}} x_{\text{av}\mathbb{D}}, \quad \text{as } p \to 0,$$
where
$$E_{\text{av}}^{\text{imp}} := \sum_{i=0}^{n-2} (d_1 (E_2^{\text{imp}})^{i+1} A_1^{\text{diff}} + d_2 (E_1^{\text{imp}})^{i+1} A_2^{\text{diff}})(A_{\text{av}}^{\text{diff}})^i.$$

(iii) In particular, for any compact $\mathfrak{I} \subseteq (0,\infty)$:
$$\boxed{x_p \xrightarrow[\mathbb{D}_{\mathfrak{I}}]{} (I - E_{\text{av}}^{\text{imp}}) x_{\text{av}\mathbb{D}}}, \quad \text{as } p \to 0$$

Proof: In [5, Thm. 2] the convergence (i) was already shown, and it remains to show the convergence of the impulsive part. First observe, that $A_i^{\text{diff}} = A_i^{\text{diff}} \Pi_i = \Pi_i A_i^{\text{diff}}$ and hence $\mathrm{e}^{A_i^{\text{diff}} d_i p} \Pi_i = \Pi_i \mathrm{e}^{A_i^{\text{diff}} d_i p} \Pi_i$, $i = 1, 2$. This implies, invoking (4), for $k > 0$,
$$x_p^f(t_k^+) = \Pi_1 x_p^f(t_k^-) = \Pi_1 \Pi_2 x_p^f(t_k^-) = \Pi_\cap x_p^f(t_k^-)$$
and, for $s_k := t_k + d_1 p$,
$$x_p^f(s_k^+) = \Pi_2 x_p^f(s_k^-) = \Pi_2 \Pi_1 x_p^f(s_k^-) = \Pi_\cap x_p^f(s_k^-),$$
i.e. at each switch the impulse-free part of the solution jumps back to the intersection $\operatorname{im} \Pi_\cap = \operatorname{im} \Pi_1 \cap \operatorname{im} \Pi_2$ of the consistency spaces (cf. [5, Lem. 1]).

Claim:
$$E_1^{\text{imp}} x_p^f(t_k^-) = d_2 p E_1^{\text{imp}} A_2^{\text{diff}} x_{\text{av}}(t_k) + O(p^2),$$
$$E_2^{\text{imp}} x_p^f(s_k^-) = d_1 p E_2^{\text{imp}} A_1^{\text{diff}} x_{\text{av}}(s_k) + O(p^2).$$

To show this claim, note first that $x_p^f(t_k^-) = \mathrm{e}^{A_2^{\text{diff}} d_2 p} x^f(s_{k-1}^+)$, hence, invoking the general approximation $e^{A\tau} = I + A\tau + O(p^2)$ for any $\tau \in [0, p]$,
$$\begin{aligned} E_1^{\text{imp}} x_p^f(t_k^-) &= E_1^{\text{imp}} (I + d_2 p A_2^{\text{diff}} + O(p^2)) x_p^f(s_{k-1}^+) \\ &= E_1^{\text{imp}} x_p^f(s_{k-1}^+) + d_2 p E_1^{\text{imp}} A_2^{\text{diff}} x_p^f(s_{k-1}^+) \\ &\quad + O(p^2). \end{aligned}$$

From $x_p^f(s_{k-1}^+) \in \operatorname{im} \Pi_\cap \subseteq \operatorname{im} \Pi_1 \subseteq \ker E_1^{\text{imp}}$ and $x_p^f(s_{k-1}^+) = x_{\text{av}}(s_{k-1}) + O(p)$ due to (i) it follows that
$$E_1^{\text{imp}} x_p^f(t_k^-) = d_2 p E_1^{\text{imp}} A_2^{\text{diff}} x_{\text{av}}(s_{k-1}) + O(p^2).$$

Finally, observing that $x_{\text{av}}(s_{k-1}) = e^{-A_{\text{av}}^{\text{diff}} d_2 p} x_{\text{av}}(t_k) = (I - d_2 p A_{\text{av}}^{\text{diff}} + O(p^2)) x_{\text{av}}(t_k) = x_{\text{av}}(t_k) + O(p)$, the claim is shown for $E_1^{\text{imp}} x_p^f(t_k^-)$; analogous argument show the claim also for $E_2^{\text{imp}} x_p^f(s_k^-)$. To show convergence of the impulse part $x_p[\cdot]$ let $\varphi \in \mathcal{C}_0^\infty$ with $\operatorname{supp} \varphi \subseteq \mathfrak{I}$, then
$$\begin{aligned} x_p[\cdot](\varphi) &= \sum_{t_k \in I} x_p[t_k](\varphi) + \sum_{s_k \in I} x_p[s_k](\varphi) \\ &= -\sum_{t_k \in I} \sum_{i=0}^{n-2} (E_1^{\text{imp}})^{i+1} x_p^f(t_k^-) \varphi^{(i)}(t_k) \\ &\quad - \sum_{s_k \in I} \sum_{i=0}^{n-2} (E_2^{\text{imp}})^{i+1} x_p^f(s_k^-) \varphi^{(i)}(s_k) \\ &= -\sum_{i=0}^{n-2} \sum_{t_k \in I} (E_1^{\text{imp}})^{i+1} A_2^{\text{diff}} d_2\, p\, x_{\text{av}}(t_k) \varphi^{(i)}(t_k) \\ &\quad - \sum_{i=0}^{n-2} \sum_{s_k \in I} (E_2^{\text{imp}})^{i+1} A_1^{\text{diff}} d_1\, p\, x_{\text{av}}(s_k) \varphi^{(i)}(s_k) \\ &\quad + \sum_{t_k, s_k \in I} O(p^2). \end{aligned}$$

Since there are at most $2(1 + (b-a)/p)$ many switchings in the compact interval $\mathfrak{I} =: [a, b]$ it follows that
$$\sum_{t_k, s_k \in I} O(p^2) = O(p).$$

Recall the properties of a (right) Riemann sum of a continuous function f on $[a, b]$:
$$\sum_{t_k \in [a,b]} p f(t_k) = \int_a^b f + O(p),$$
hence, for any $i = 0, \ldots, n-2$:
$$\begin{aligned} &\sum_{t_k \in I} (E_1^{\text{imp}})^{i+1} A_2^{\text{diff}} d_2\, p\, x_{\text{av}}(t_k) \varphi^{(i)}(t_k) \\ &\qquad = (E_1^{\text{imp}})^{i+1} A_2^{\text{diff}} d_2 \int_a^b x_{\text{av}} \varphi^{(i)} + O(p). \end{aligned}$$

Furthermore, invoking $\varphi^{(i)}(a) = 0 = \varphi^{(i)}(b)$ and partial integration it follows that
$$\int_a^b x_{\text{av}} \varphi^{(i)} = \int_a^b x_{\text{av}}^{(i)} \varphi = \int_a^b (A_{\text{av}}^{\text{diff}})^i x_{\text{av}} \varphi,$$
hence (using analogous arguments for the impulses at s_k)
$$\begin{aligned} x_p[\cdot](\varphi) &= \sum_{i=0}^{n-2} (E_1^{\text{imp}})^{i+1} A_2^{\text{diff}} d_2 (A_{\text{av}}^{\text{diff}})^i \int_a^b x_{\text{av}} \varphi \\ &\qquad + \sum_{i=0}^{n-2} (E_2^{\text{imp}})^{i+1} A_1^{\text{diff}} d_1 (A_{\text{av}}^{\text{diff}})^i \int_a^b x_{\text{av}} \varphi \\ &\qquad + O(p) \\ &= -E_{\text{av}}^{\text{imp}} \int_a^b x_{\text{av}} \varphi + O(p) \\ &= -E_{\text{av}}^{\text{imp}} x_{\text{av}\mathbb{D}}(\varphi) + O(p). \end{aligned}$$

In particular, $x_p[\cdot](\varphi) \to -E_{\text{av}}^{\text{imp}} x_{\text{av}\mathbb{D}}(\varphi)$ as $p \to 0$, which concludes the proof.

■

For the example from Section III the involved matrices are given as follows:
$$\Pi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \Pi_2 = I$$
which clearly commute, hence Theorem 1 is applicable. Furthermore,
$$A_1^{\text{diff}} = 0, \quad A_2^{\text{diff}} = A_2, \quad A_{\text{av}}^{\text{diff}} = 0,$$
and
$$E_1^{\text{imp}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_2^{\text{imp}} = 0, \quad E_{\text{av}}^{\text{imp}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d_2 & 0 & 0 \end{bmatrix}.$$

Hence the function part x_p^f of the solution of the switched DAE converges to the constant trajectory

$$x_{\text{av}}(t) = (x_0^0, 0, 0)^\top$$

and the impulsive part $x_p[\cdot]$ converges to the constant function

$$-E_{\text{av}}^{\text{imp}} x_{\text{av}}(t) = (0, 0, -d_2 x_1^0)^\top$$

which coincides with the ad hoc analysis from Section III.

Theorem 1 also provides a simple criteria which ensures that the Dirac impulses have no influence on the averaging:

Corollary 2: With the same notation as in Theorem 1 assume:

$$E_2^{\text{imp}} A_1^{\text{diff}} = 0 \quad \text{and} \quad E_1^{\text{imp}} A_2^{\text{diff}} = 0.$$

Then, for any compact interval $\mathfrak{I} \in (0, \infty)$,

$$x_p \underset{\mathbb{D}_{\mathfrak{I}}}{\longrightarrow} x_{\text{av}\mathbb{D}}.$$

Remark 3 (Commuting consistency projectors): The crucial assumption of Theorem 1 is commutativity of the consistency projectors. This assumption was also used (and motivated) in [5] to show convergence of the impulse-free part of the solution, but was recently relaxed [8]. However, it is not clear whether this relaxation is also applicable for the convergence of the impulsive part of the solution. As already pointed out in [5, Rem. 4] a straight-forward generalization of the averaging result to more than two modes is not possible, but with a different proof technique (e.g. the one used in [4] or [8]) it may be possible to also show convergence of the impulsive part for more than two modes. In the special case of commuting A^{diff} matrices it follows that all consistency projectors commute with each other as well as with all A^{diff}-matrices [6, Lem. 9]; hence in that case the proof technique used here can be generalized to more than two modes.

V. Conclusion

It was shown that the presence of Dirac impulses in the solution of switched DAEs do not prevent applying the well established averaging technique. However, the Dirac impulses cannot be neglected, although the limit is impulse free. This effect is also visible when considering the more realistic scenario where instead of ideal Dirac impulses the approximation of Dirac impulses is considered. Hence one can expect that this theoretical result will also play an important role when applying the average technique in the real world.

This note only considers the case of two modes with commuting consistency projectors and the author believes that a generalization to more than two modes and non-commuting consistency projectors should be possible. However, the proof technique used here is only possible for two modes with commuting projectors, because it was utilized that after each switch the trajectory jumps back into the intersection of the consistency spaces. This is not true anymore for switched systems with more than two modes, hence another approach is needed and is the topic of future research.

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