A normal form for pure differential algebraic systems

Stephan Trenn

Institute for Mathematics, Ilmenau University of Technology, Ilmenau, Germany

Abstract

In this paper linear time-invariant differential algebraic equations (DAEs) are studied; the focus is on pure DAEs which are DAEs without an ordinary differential equation (ODE) part. A normal form for pure DAEs is given which is similar to the Byrnes-Isidori normal form for ODEs. Furthermore, the normal form exhibits a Kalman-like decomposition into impulse-controllable- and impulse-observable states. This leads to a characterization of impulse-controllability and -observability.

Key words:
differential algebraic equation, normal form, impulse controllability, impulse observability
2000 MSC: 34A09, 34C20

1. Introduction

Differential algebraic equations (DAEs) of the form

$$\begin{aligned} E\dot{x} &= Ax + bu, \\ y &= cx, \end{aligned} \tag{1}$$

$(c, E, A, b) \in \mathbb{R}^{1\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^n$, $n \in \mathbb{N}$, play an important role in systems theory. Those equations arise when modelling for example electrical circuits, mechanical systems, or, in general, dynamical systems with additional algebraic conditions. Interconnected ordinary differential equations (ODEs) can also be described as a DAE. There is a wide range of literature for DAEs of the form (1), i.e. linear time-invariant DAEs, e.g. [1, 2, 3, 4, 5].

Email addresses: `stephan.trenn@tu-ilmenau.de` (Stephan Trenn)

Normal or condensed forms for DAEs have always been a research topic and the most famous normal form is the Kronecker normal form or, if one considers a special class of DAEs, the Weierstraß normal form. The latter is basically a decoupling into an ODE and a "pure" DAE. Most normal or condensed forms concentrate on the two matrices E and A and not on the input and output vectors b and c. But for control problems normal forms must incorporate the input and output. For ODEs the Byrnes-Isidori normal form (which focus on the relative degree [6, p. 165], see also [7, Lem. 3.5]) and the Kalman-decomposition (which focus on controllable and observable sub-states [8]) are examples of such normal forms.
This paper gives a normal form for "pure" DAEs which can be seen as a generalization of the Byrnes-Isidori normal form combined with a Kalman-like decomposition. In fact, the state space is separated into impulse-controllable and -observable sub-states, see Theorem 24. Compared to a similar decomposition proposed in [3, p. 52] (without proof) the normal form from Theorem 20 is more specific and allows for a better analysis.
There are already results on normal or condensed forms of DAEs available, e.g. [9], [10], [11], [5]. But none of these result focus on the relative degree or on impulse-controllable and -observable states. In addition they partly use a different concept of equivalence which leads to other normal forms. On the other hand some of these results go much further as the results in this paper because rectangular (in particular non-regular) DAEs with time-varying coefficients are considered.

This paper is structured as follows. First, some preliminaries (Section 2) are given, in particular the subtle difference between DAEs and differential algebraic systems (DASs) is explained. Section 3 deals with the transfer function of DASs and realization theory, in particular some specific minimal realizations of pure DASs are given. Before stating the main results in Section 5, impulse-controllability and -observability are revisited in Section 4, the invariants impulse-controllability-index and impulse-observability-index are defined. The main result is the normal form given in Theorem 20. This normal form can be used to give new characterizations of impulse-controllability and -observability, see Theorem 24.

The following notation will be used throughout this paper. $\mathbb{N}$ and $\mathbb{R}$ are the natural and real numbers, $\mathbb{R}[s]$ is the ring of polynomials and $\mathbb{R}(s)$ is the field of rational functions with real coefficients. For a polynomial $p(s) \in \mathbb{R}[s]$ the degree of $p(s)$ is denoted by $\deg p(s)$. The matrix $I \in \mathbb{R}^{n\times n}$

is the identity matrix of size $n \in \mathbb{N}$, the latter is in general clear from the context. For two square matrizes A,B let $\operatorname{diag}(A, B) := \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. For the row vectors $c_1, c_2, \ldots, c_n \in \mathbb{R}^{1\times m}$, $n \in \mathbb{N}$, of the same length $m \in \mathbb{N}$ let $[c_1/c_2/\ldots/c_n] := \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^{n\times m}$. The rank, image, and kernel of matrix A is denoted by $\operatorname{rk} A$, $\operatorname{im} A$, and $\ker A$, resp.

2. Preliminaries: Differential algebraic systems (DASs)

In this work only differential algebraic *systems* (DASs), i.e. matrix-tuples (see Definition 1), are considered and not differential algebraic *equations* (DAEs) like (1). The reason is that for the latter one has always to define what the variable x should be. In particular it would be necessary to specify an appropriate solution space. Since the results of this work are independent of the chosen solution space, any discussion about solution spaces will be avoided by considering DASs instead of DAEs.

Definition 1 (DASs, regular and pure DASs, ODSs). *A* differential algebraic system (DAS) *with state space dimension* $n \in \mathbb{N}$ *is a tupel* $(c, E, A, b) \in \mathbb{R}^{1\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^n$. *The space of all DASs with state space dimension* n *is*

$$\Sigma_n := \mathbb{R}^{1\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^n.$$

The space of regular *DASs with state space dimension* $n \in \mathbb{N}$ *is*

$$\Sigma_n^{\mathrm{reg}} := \{ \ (c, E, A, b) \in \Sigma_n \ \mid \det(Es - A) \in \mathbb{R}[s]\backslash\{0\} \ \}.$$

The space of pure *DASs is*

$$\Sigma_n^{\mathrm{pure}} := \left\{ \ (c, E, A, b) \in \Sigma_n \ \middle| \ \det A \neq 0 \ \wedge \ A^{-1}E \textit{ is nilpotent } \right\}.$$

The space of ordinary differential systems (ODSs) *is*

$$\Sigma_n^{\mathrm{ODS}} := \{ \ (c, E, A, b) \in \Sigma_n \ \mid \det E \neq 0 \ \}.$$

Remark 2. (i) *Every pure DAS and every ODS is regular, i.e. for all* $n \in \mathbb{N}$

$$\Sigma_n^{\mathrm{pure}} \subseteq \Sigma_n^{\mathrm{reg}} \quad \textit{and} \quad \Sigma_n^{\mathrm{ODS}} \subseteq \Sigma_n^{\mathrm{reg}}.$$

(ii) *No pure DAS is an ODS and vice versa, i.e. for all* $n \in \mathbb{N}$

$$\Sigma_n^{\text{pure}} \cap \Sigma_n^{\text{ODS}} = \emptyset.$$

(iii) *For invertible* $A \in \mathbb{R}^{n\times n}$ *and some* $E \in \mathbb{R}^{n\times n}$, $n \in \mathbb{N}$, *the matrix* $A^{-1}E$ *is nilpotent if, and only if,* EA^{-1} *is nilpotent, hence*

$$\Sigma_n^{\text{pure}} = \left\{ \ (c, E, A, b) \in \Sigma_n \ \middle| \ \det A \neq 0 \ \wedge \ EA^{-1} \text{ is nilpotent } \right\}.$$

Definition 3 (Equivalence). *Two DASs* $(c_1, E_1, A_1, b_1) \in \Sigma_{n_1}$ *and* $(c_2, E_2, A_2, b_2) \in \Sigma_{n_2}$, $n_1, n_2 \in \mathbb{N}$, *are called* equivalent, *written*

$$(c_1, E_1, A_1, b_1) \simeq (c_2, E_2, A_2, b_2),$$

if, and only if, $n_1 = n_2 =: n$ *and there exist invertible matrices* $S, T \in \mathbb{R}^{n\times n}$ *such that*

$$(c_2, E_2, A_2, b_2) = (c_1T, SE_1T, SA_1T, Sb_1).$$

Note that $\simeq$ is an equivalence relation.

Remark 4. *Every regular DAS* (c, E, A, b) *is equivalent to a DAS in* Weierstrass form

$$\left([\ c_1 \mid c_2\ ], \left[\begin{array}{c|c} I & 0 \\ \hline 0 & N \end{array}\right], \left[\begin{array}{c|c} J & 0 \\ \hline 0 & I \end{array}\right], \left[\begin{array}{c} b_1 \\ \hline b_2 \end{array}\right] \right)$$

where N *is a nilpotent matrix ([12], see also [5, Thm. 2.7]). Clearly, this is a (unique) decomposition into an ODS (also called slow system) and a pure DAS (also known as the fast system).*

Proposition 5. *For all* $n \in \mathbb{N}$

$$\Sigma_n^{\text{pure}} = \left\{ \ (c, E, A, b) \in \Sigma_n \ \middle| \ \exists (\hat{c}, N, I, \hat{b}) \in \Sigma_n^{\text{pure}} : \ (c, E, A, b) \simeq (\hat{c}, N, I, \hat{b}) \ \right\}.$$

Proof. By definition, every pure DAE $(c, E, A, b) \in \Sigma_n$ is equivalent to $(c, A^{-1}E, I, A^{-1}b) = (\hat{c}, N, I, \hat{b}) \in \Sigma_n^{\text{pure}}$. If $(c, E, A, b) \in \Sigma_n$ is equivalent to $(\hat{c}, N, I, \hat{b}) \in \Sigma_n^{\text{pure}}$ then there exist invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that $A = SIT = ST$ and $E = SNT$. In particular, A is invertible and $A^{-1}E = T^{-1}NT$. By assumption, N is nilpotent and hence $A^{-1}E$ is nilpotent which implies that (c, E, A, b) is pure.

qed

Proposition 5 (see also the forthcoming Proposition 15) justifies that every pure DASs can be considered to be in the *standard* form (c, N, I, b), where N is a nilpotent matrix.

3. Transfer function and minimal realization

In this section transfer functions of DASs and minimal realizations are studied. From the theory of ODEs it is well known, that the transfer function is a useful tool to study the input-output behaviour of a linear system. Furthermore the definition of the (negative) relative degree is based on transfer functions and the negative relative degree is important for the normal form of pure DASs given in this paper (Theorem 20). It will also turn out that one of the given minimal realization is a "part" of this normal form.

Definition 6 (Transfer function and IO-equivalence). *The* transfer function *of a regular DAS* $(c, E, A, b) \in \Sigma_n^{\text{reg}}$*,* $n \in \mathbb{N}$*, is the rational function* $g(s) \in \mathbb{R}(s)$ *given by*

$$g(s) = c(Es - A)^{-1}b.$$

Two regular DASs $(c_1, E_1, A_1, b_1) \in \Sigma_{n_1}^{\text{reg}}$*,* $n_1 \in \mathbb{N}$*, and* $(c_2, E_2, A_2, b_2) \in \Sigma_{n_2}^{\text{reg}}$*,* $n_2 \in \mathbb{N}$*, are called* IO-equivalent*, written*

$$(c_1, E_1, A_1, b_1) \sim (c_2, E_2, A_2, b_2),$$

if, and only if, their corresponding transfer functions $g_1(s)$ *and* $g_2(s)$ *are equal.*

Remark 7. *For two regular DASs* $(c_1, E_1, A_1, b_1) \in \Sigma_{n_1}^{\text{reg}}$*,* $n_1 \in \mathbb{N}$*, and* $(c_2, E_2, A_2, b_2) \in \Sigma_{n_2}^{\text{reg}}$*,* $n_2 \in \mathbb{N}$ *it follows from the definition that*

$$(c_1, E_1, A_1, b_1) \simeq (c_2, E_2, A_2, b_2) \quad \Rightarrow \quad (c_1, E_1, A_1, b_1) \sim (c_2, E_2, A_2, b_2).$$

The converse is in general not true, but this question is strongly related to minimal realizations, impulse-controllability and (impulse-)observability, which are studied later in this work (see Corollary 26 and Remark 27).

Remark 8. (i) *The transfer function of an ODS is strictly proper (i.e. the degree of the numerator is smaller than the degree of the denominator).*

(ii) *The transfer function of a pure DAS is a polynomial, in particular if the pure DAS is in standard form* $(c, N, I, b) \in \Sigma_n^{\text{pure}}$*,* $n \in \mathbb{N}$*, the transfer function is given by*

$$g(s) = -\sum_{i=0}^{n-1} cN^i b s^i.$$

For convenience, if $n = 0$ *the transfer function is defined as* $g(s) \equiv 0$*.*

Definition 9. *A* realization *of a transfer function* $g(s) \in \mathbb{R}(s)$ *is a regular DAS* $(c, E, A, b) \in \Sigma_n^{\text{reg}}$*,* $n \in \mathbb{N}$*, with* $g(s) = c(Es - A)^{-1}b$*. A realization is called* minimal *if, and only if, there exists no other realization with smaller state-space dimension.*

For a given transfer function it is an interesting question how a realization might look, what the minimal dimension is and if there are some standard realizations. For ODEs these questions are studied in realization theory and most relevant questions are answered. The next propositions shows that for the realization theory of DASs one can basically concentrate on pure DASs.

Proposition 10. *Let* $g(s) = \frac{p(s)}{q(s)} \in \mathbb{R}(s)$*, then there exists unique* $p_1(s), p_2(s) \in \mathbb{R}[s]$ *with* $\deg p_2(s) < \deg q(s)$ *such that* $g(s) = p_1(s) + \frac{p_2(s)}{q(s)}$*.*
If $(c_1, E_1, A_1, b_1) \in \Sigma_{n_1}^{\text{pure}}$*,* $n_1 \in \mathbb{N}$ *and* $(c_2, E_2, A_2, b_2) \in \Sigma_{n_2}^{\text{ODS}}$*,* $n_2 \in \mathbb{N}$ *are realizations of* $p_1(s)$ *and* $\frac{p_2(s)}{q(s)}$*, resp., then*

$$(c, E, A, b) := \big((c_1, c_2), \operatorname{diag}(E_1, E_2), \operatorname{diag}(A_1, A_2), (b_1^\top, b_2^\top)^\top\big) \in \Sigma_{n_1+n_2}^{\text{reg}}$$

is a realization of $g(s)$*.*
Conversely, every realization of $g(s)$ *is equivalent to*

$$(c, E, A, b) = \big((c_1, c_2), \operatorname{diag}(E_1, E_2), \operatorname{diag}(A_1, A_2), (b_1^\top, b_2^\top)^\top\big) \in \Sigma_{n_1+n_2}^{\text{reg}},$$

where $(c_1, E_1, A_1, b_1) \in \Sigma_{n_1}^{\text{pure}}$*,* $n_1 \in \mathbb{N}$*, is a pure realization of* $p_1(s)$ *and* $(c_2, E_2, A_2, b_2) \in \Sigma_{n_2}^{\text{ODS}}$*,* $n_2 \in \mathbb{N}$*, is an ODS realizations of* $\frac{p_2(s)}{q(s)}$ *(*$n_1 = 0$ *or* $n_2 = 0$ *is possible).*
Furthermore, in both cases (c_i, E_i, A_i, b_i)*,* $i = 1, 2$*, are minimal realizations if, and only if,* (c, E, A, b) *is a minimal realization.*

Proof. The unique decomposition $g(s) = p_1(s) + \frac{p_2(s)}{g(s)}$ with $\deg p_2(s) < \deg q(s)$ is a well known algebraic result (Euclidian algorithm for polynomials). From the definition of the transfer function it easily follows that (c, E, A, b) is a realization of $g(s)$ if (c_1, E_1, A_1, b_1) and (c_2, E_2, A_2, b_2) are realizations of $p_1(s)$ and $\frac{p_2(s)}{q(s)}$, resp. Conversely, observe that every regular DAS is equivalent to a DAS in Weierstraß form (see Remark 4), which yields the assertion of the proposition. It remains to show the minimality property. Clearly, if (c, E, A, b) is minimal then (c_i, E_i, A_i, b_i), $i = 1, 2$, are minimal, too. To show that (c, E, A, b) is minimal if (c_i, E_i, A_i, b_i), $i = 1, 2$, are minimal, consider any realization $(\hat{c}, \hat{E}, \hat{A}, \hat{c}) \in \Sigma_{\hat{n}}^{\text{reg}}$ with $\hat{n} \in \mathbb{N}$. Then this realization is equivalent to a DAS in Weierstraß form with dimension $\hat{n}$, whose pure DAS and ODS parts have dimensions $\hat{n}_1 \in \mathbb{N}$ and $\hat{n}_2 \in \mathbb{N}$, resp. Let the transfer function of the pure DAS part be the polynomial $\hat{p}_1(s) \in \mathbb{R}[s]$ and let the transfer function of the ODS part be the strictly proper rational function $\frac{\hat{p}_2(s)}{\hat{q}(s)} \in \mathbb{R}(s)$. Since $\hat{p}_1(s) + \frac{\hat{p}_2(s)}{\hat{q}(s)} = g(s) = p_1(s) + \frac{p_2}{q(s)}$ it follows that $\hat{p}_1(s) = p_1(s)$ and $\frac{\hat{p}_2(s)}{\hat{q}(s)} = \frac{p_2(s)}{q(s)}$. This implies, by the minimality assumption, that $\hat{n}_i \geq n_i$, $i = 1, 2$, hence $\hat{n} = \hat{n}_1 + \hat{n}_2 \geq n_1 + n_2$, i.e. the given realization of $g(s)$ with dimension $n_1 + n_2$ is minimal.

$\boxed{\text{qed}}$

The foregoing proposition justifies that for a realization theory of general DASs it is sufficient to consider pure DASs and ODSs separately. Realization theory of ODSs is well understood, hence it remains to study the realization theory of pure DASs.

Proposition 11. *Consider a polynomial transfer function $g(s) = \sum_{i=0}^{r} \alpha_i s^i \in \mathbb{R}[s]$ for $r \in \mathbb{N}$, $\alpha_r \neq 0$. Then the following DASs are minimal realizations of $g(s)$ with state space dimension $r + 1$:*

(i) R-form*:*

$$\left([0, \ldots, 0, 1], \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & \ddots \\ & & 1 \quad 0 \end{bmatrix}, I_*, \begin{bmatrix} -\alpha_r \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) \in \Sigma_{r+1}^{\text{pure}},$$

where

$$I_* = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ \frac{-\alpha_0}{\alpha_r} & \cdots & \frac{-\alpha_{r-1}}{\alpha_r} & 1 \end{bmatrix}.$$

(ii) O-form*:*

$$\left([0, \ldots, 0, 1], \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & \ddots \\ & & 1 \quad 0 \end{bmatrix}, I, \begin{bmatrix} -\alpha_r \\ \vdots \\ -\alpha_0 \end{bmatrix} \right) \in \Sigma_{r+1}^{\text{pure}}.$$

(iii) C-form*:*

$$\left([-\alpha_0, \ldots, -\alpha_r], \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & \ddots \\ & & 1 \quad 0 \end{bmatrix}, I, \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) \in \Sigma_{r+1}^{\text{pure}}.$$

Proof. A simple calculation invoking Remark 8 (ii) shows that the tranfer function of the O- and C-form coincide with $g(s)$. With $S = -\alpha_r I$ and $T = -\frac{1}{\alpha_r} I_*$ the C-form and the R-form are equivalent and hence the transfer function of the R-form also coincides with $g(s)$.
To show that these realizations are minimal consider any realization $(c, E, A, b) \in \Sigma_n^{\text{reg}}$, $n \in \mathbb{N}$, of $g(s)$. The ODS part and pure DAS part of the Weierstraß form (see Remark 4) have state dimensions $n_1, n_2 \in \mathbb{N}$ with $n_1+n_2 = n$ and transfer functions $g_1(s) \in \mathbb{R}(s)$ and $g_2(s) \in \mathbb{R}[s]$, resp. By Remark 8, $g_1(s)$ is strictly proper and $\deg g_2(s) \leq n_2 - 1$. Together with $g_1(s) + g_2(s) = g(s) \in \mathbb{R}[s]$ this yields $g_1(s) \equiv 0$ and $r = \deg g(s) = \deg g_2(s) \leq n_2 - 1$. This shows $n = n_1 + n_2 \geq r + 1$, i.e. the minimal state-space dimension for any realization of $g(s)$ is $r + 1$.

$\boxed{\text{qed}}$

First results on realization theory can be found in [3], in particular Proposition 11 is a constructive version of [3, Lemma 2-6.2] and Proposition 10 is implicitely used in the proof of [3, Thm. 2-6.3].

4. Impulse-controllability and impulse-observability

From the theory of linear ODEs it is well known, that the controllability- and observability-matrices play an important role for controllability and observability as well as for the construction of normal forms. It is possible to define analogous matrices for DASs, which play similar roles. Furthermore one can define impulse-controllability- and impulse-observability-indices which are invariants with respect to equivalence transformations. This is important for the normal form and can be used for characterizations of impulse-controllability and -observability.

Definition 12. *Consider a pure DAS* $(c, E, A, b) \in \Sigma_n^{\text{pure}}$, $n \in \mathbb{N}$. *The* impulse-controllability-matrix *of* (c, E, A, b) *is*

$$\mathcal{B}_{\text{imp}} := \left[b, N_b b, {N_b}^2 b, \ldots, {N_b}^{n-1} b\right], \quad \textit{where } N_b := EA^{-1}.$$

The impulse-controllability-index *of* (c, E, A, b) *is*

$$d_b := \operatorname{rk} \mathcal{B}_{\text{imp}}.$$

The impulse-observability-matrix *of* (c, E, A, b) *is*

$$\mathcal{C}_{\text{imp}} := \left[c / cN_c / cN_c{}^2 / \dots / cN_c{}^{n-1}\right], \quad \textit{where } N_c := A^{-1}E.$$

The impulse-observability-index *of* (c, E, A, b) *is*

$$d_c := \operatorname{rk} \mathcal{C}_{\text{imp}}.$$

Definition 13. *A pure DAS* $(c, E, A, b) \in \Sigma_n^{\text{pure}}$*,* $n \in \mathbb{N}$*, is called*

(i) impulse-controllable *(in the sense of [13]) if, and only if,*

$$\operatorname{im} N_b = \operatorname{im}(N_b \mathcal{B}_{\text{imp}}), \quad \textit{where } N_b = EA^{-1}.$$

(ii) impulse-observable *(in the sense of [13]) if, and only if,*

$$\ker N_c = \ker(\mathcal{C}_{\text{imp}} N_c), \quad \textit{where } N_c = A^{-1}E.$$

Remark 14. *It might seem artificial to define impulse-controllability and -observability in terms of algebraic conditions. A natural definition should be based on reachability of certain "impulsive" states and deduction of "impulsive" states from the output. The problem is that these definitions would require a complete distributional solution theory leading to an unnecessary overhead for the purposes of this paper. For this reason, the definition of impulse-controllability and -observability is based on characterizations given in [13, Thm. 4 and Thm. 9].*

Proposition 15. *Impulse-controllability and -observability as well as the corresponding indices are invariant under equivalence transformations.*

Proof. Let $(c_1, E_1, A_1, b_1), (c_2, E_2, A_2, b_2) \in \Sigma_n^{\text{pure}}$ be equivalent via $S, T \in \mathbb{R}^{n \times n}$. Let $\mathcal{B}_{\text{imp},1}$, $\mathcal{C}_{\text{imp},1}$, $\mathcal{B}_{\text{imp},2}$, and $\mathcal{C}_{\text{imp},2}$ be the corresponding impulse-controllabel- and impulse-observable-matrices. From the definition it follows that

$$\mathcal{B}_{\text{imp},2} = S \mathcal{B}_{\text{imp},1},$$

hence the corresponding inpulse-controlability-indices are equal. Furthermore,

$$\operatorname{im} E_2 A_2{}^{-1} = \operatorname{im} S E_1 A_1{}^{-1} S^{-1} = \operatorname{im} S E_1 A_1{}^{-1},$$

which yields that (c_1, E_1, A_1, b_1) is impulse-controllable if, and only if, (c_2, E_2, A_2, b_2) is impulse-controllable. Analogously,

$$\mathcal{C}_{\text{imp},2} = \mathcal{C}_{\text{imp},1} T$$

and

$$\ker {A_2}^{-1} E_2 = \ker T^{-1} {A_1}^{-1} E_1 T = \ker {A_1}^{-1} E_1 T,$$

which show that the impulse-observability-index and impulse-controllability are invariant.

qed

Proposition 15 again justifies that one can assume that every pure DAS is in the standard form (c, N, I, b). In fact, this simplifies the Definitions 12 and 13 because then $N_b = N_c = N$.

The next proposition highlights an important property of the impulse-controllability- and impulse-observability-matrices.

Proposition 16. *Consider a pure DAS in standard form $(c, N, I, b) \in \Sigma_n^{\text{pure}}$, $n \in \mathbb{N}$, with impulse-controllability- and impulse-observability-indices $d_b, d_c \in \mathbb{N}$. Then*

$$\mathcal{B}_{\text{imp}} = \left[b, Nb, \ldots, N^{d_b - 1} b, 0, \ldots, 0\right]$$

and

$$\mathcal{C}_{\text{imp}} = \left[c / cN / \ldots / cN^{d_c - 1} / 0 / \ldots / 0\right].$$

Proof. Let $d \in \mathbb{N}$ be the smallest number such that $N^d b = 0$ (which exists since N is nilpotent). In terms of [14, XII.7] the vector b is N-cyclic with period d. Now [14, Lemma XII.7.1] states that $[b, Nb, \ldots, N^{d-1} b]$ has full rank which yields

$$d_b = \operatorname{rk} \mathcal{B}_{\text{imp}} = \operatorname{rk} \left[b, Nb, \ldots, N^{d-1} b, 0, \ldots, 0\right] = d,$$

this is the assertion of the proposition. The same argument applied to $N^\top$ and $c^\top$ shows the analogous property for $\mathcal{C}_{\text{imp}}$.

qed

Remark 17. *For $(c, E, A, b) \in \Sigma_n^{\text{pure}}$ let $(b^\top, E^\top, A^\top, c^\top) \in \Sigma_n^{\text{pure}}$ be the* dual system *(see e.g. [3, 2.4]). If d_b and d_c are the impulse-controllable- and impulse-observable-indices of (c, E, A, b), then it is easy to see that d_c and d_b are the impulse-controllable- and impulse-observable-indices of the dual system.*

5. A normal form for DASs

In this section a normal form for pure DASs is given. For the derivation of the normal form the following definition of the negative relative degree is needed.

Definition 18. *Consider a pure DAS* $(c, E, A, b) \in \Sigma_n^{\text{pure}}$*,* $n \in \mathbb{N}$*, with (polynomial) transfer function* $g(s) \in \mathbb{R}[s]$*. The* negative relative degree *of* (c, E, A, b) *is*

$$r := \deg g(s).$$

By convention, if $g(s) \equiv 0$ *then* $r := -\infty$*.*

Remark 19. (i) *For an ODS with transfer function* $g(s) = \frac{p(s)}{q(s)}$ *the* relative degree ρ *is defined as the difference between the degrees of the denominator and numerator, i.e.* $\rho := \deg q(s) - \deg p(s)$*. This definition is consistent with Definition 18 and* $r = -\rho$*.*

(ii) *By Remark 7 the negative relative degree is invariant under equivalence transformations. Furthermore, for a pure DAS in standard form* $(c, N, I, b) \in \Sigma_n^{\text{pure}}$ *the negative relative degree fulfills (see Remark 8 (ii))*

$$r = \max\left\{ \, i \in \mathbb{N} \;\middle|\; cN^i b \neq 0 \, \right\},$$

where by convention the maximum of an empty set is $-\infty$*.*

It is now possible to formulate the main result of this paper. With the proposed normal form, the influence of the input on the states and the influence of states on the output can easily be seen.

Theorem 20. *Consider a pure DAS* $(c, E, A, b) \in \Sigma_n^{\text{pure}}$*,* $n \in \mathbb{N}$*, with negative relative degree* $r \geq 0$*, impulse-controllability- and impulse-observability-indices* $d_b, d_c \in \mathbb{N}$*. Then* (c, E, A, b) *is equivalent to* $(\hat{c}, \hat{N}, \hat{I}, \hat{b}) \in \Sigma_n^{\text{pure}}$*,* *where*

$$\hat{c} = [0,\dots,0,1], \quad \hat{N} = \begin{array}{c}\left[\begin{array}{c|c|c|c} \begin{smallmatrix}0&&\\1&\ddots&\\&1&0\end{smallmatrix} & 0 & 0 & 0 \\ \hline E_1 & N_1 & 0 & 0 \\ \hline E_2 & E_3 & \begin{smallmatrix}0&&\\1&\ddots&\\&1&0\end{smallmatrix} & 0^* \\ \hline 0_* & 0 & 0 & \begin{smallmatrix}0&&\\1&\ddots&\\&1&0\end{smallmatrix} \end{array}\right]\end{array} \begin{array}{l} \} \; d_c - r - 1 \\ \} \; n - d_c - d_b + r + 1 \\ \} \; d_b - r - 1 \\ \} \; r + 1 \end{array}$$

$$\hat{I} = \left[\begin{array}{c|c|c|c} I & & & \\ \hline & I & & \\ \hline & & I & \\ \hline & & & I_* \end{array}\right], \quad \hat{b} = \left[\begin{array}{c} 0 \\ \hline 0 \\ \hline 0 \\ \hline \begin{smallmatrix}\gamma\\0\\|\\0\end{smallmatrix} \end{array}\right],$$

where $\gamma := cA^{-1}(EA^{-1})^r b = c(A^{-1}E)^r A^{-1}b \neq 0,$

$$0^* = \begin{bmatrix} * - * \; 1 \\ 0 \end{bmatrix}, \quad 0_* = \begin{bmatrix} 0 & \begin{smallmatrix}1\\ *\\ |\\ *\end{smallmatrix} \end{bmatrix}, \quad I_* = \begin{bmatrix} 1 & & \\ & \ddots & \\ * - * & & 1 \end{bmatrix},$$

and $N_1 \in \mathbb{R}^{(n-d_c-d_b+r+1)\times(n-d_c-d_b+r+1)}$ *is a nilpotent matrix (in Jordan normal form).*

Proof. Without loss of generality, assume that the DAS is in standard form, i.e. $(c,E,A,b) = (c,N,I,b)$ for some nilpotent matrix N. In this case $\gamma = cN^r b \neq 0$ (see Remark 19 (ii)).

The proof consists of two main steps. The first step is the construction of the transformation matrices S and T, in particular the construction must ensure that S and T are invertible. In the second step it is shown that indeed $(c,N,I,b) \simeq (\hat{c},\hat{N},\hat{I},\hat{b})$ via S and T.

Step 1.
The construction is based on the five matrices $\mathcal{L} \in \mathbb{R}^{n\times(d_c-r-1)}$, $\overline{\mathcal{L}} \in \mathbb{R}^{n\times(n-d_c-d_b+r+1)}$,

$\overline{\mathcal{B}} \in \mathbb{R}^{n\times(d_b-r-1)}$, $\mathcal{B} \in \mathbb{R}^{n\times(r+1)}$, and $\hat{I} \in \mathbb{R}^{n\times n}$, which define the transformation matrix S and T by

$$S := \gamma\ \left[\mathcal{L}, \overline{\mathcal{L}}, \overline{\mathcal{B}}, \mathcal{B}\right]^{-1},$$
$$T := \frac{1}{\gamma}\left[\mathcal{L}, \overline{\mathcal{L}}, \overline{\mathcal{B}}, \mathcal{B}\right]\ \hat{I}.$$

Step 1a: The matrix $\hat{I}$.
Let

$$\hat{I} := \left[\begin{array}{c|c} I & \\ \hline & I_* \end{array}\right] \in \mathbb{R}^{n\times n},$$

where

$$I_* := \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & \ddots & \\ \frac{cb}{-\gamma} & \frac{cNb}{-\gamma} & \cdots & \frac{cN^{r-1}b}{-\gamma} & 1 \end{bmatrix} \in \mathbb{R}^{(r+1)\times(r+1)}. \tag{2}$$

Obviously, $\hat{I}$ is invertible.

Step 1b: The matrices $\mathcal{B}$ and $\overline{\mathcal{B}}$.
Let

$$\mathcal{B} := \left[b, Nb, \ldots, N^r b\right] \in \mathbb{R}^{n\times(r+1)}$$

and

$$\mathcal{C} := \left[cN^r / \ldots / cN / c\right] \in \mathbb{R}^{(r+1)\times n}$$

By Remark 19 (ii)

$$\mathcal{CB} = \begin{bmatrix} cN^r b & & & \\ cN^{r-1}b & \ddots & & \\ \vdots & \ddots & \ddots & \\ cb & \cdots & cN^{r-1}b & cN^r b \end{bmatrix} \in \mathbb{R}^{(r+1)\times(r+1)} \tag{3}$$

is invertible and hence $\mathcal{B}$ and $\mathcal{C}$ must have full rank. In particular this implies $d_b \geq r+1$ and $d_c \geq r+1$. Let

$$\overline{\mathcal{B}} := \left[N^{r+1}b, N^{r+2}b, \ldots, N^{d_b-1}b\right] \in \mathbb{R}^{n\times(d_b-r-1)},$$

then by the definition of d_b the matrix $\left[\overline{\mathcal{B}}, \mathcal{B}\right]$ has full column rank.

Step 1c: The matrix $\mathcal{L}$.
If $d_c = r+1$, then $\mathcal{L}$ is the empty matrix. Otherwise let

$$\overline{\mathcal{C}} := \left[cN^{d_c-1}/cN^{d_c-2}/\cdots/cN^{r+1}\right] \in \mathbb{R}^{(d_c-r-1)\times n}.$$

Then $\ker\left[\overline{\mathcal{C}}/\mathcal{C}\right]$ is an $(n-d_c)$-dimensional subspace of $\ker\mathcal{C}$ (where $\dim\ker\mathcal{C} = n-r-1$), i.e. there exists a full rank matrix $L \in \mathbb{R}^{n\times(d_c-r-1)}$ such that $\operatorname{im} L \oplus \ker\left[\overline{\mathcal{C}}/\mathcal{C}\right] = \ker\mathcal{C}$. In particular $\operatorname{im} L \cap \ker\overline{\mathcal{C}} = \{0\}$ and $\operatorname{im} L \subseteq \ker\mathcal{C}$. Let

$$\mathcal{L} := \gamma L\left(\overline{\mathcal{C}}L\right)^{-1}.$$

It remains to show that, firstly, $\mathcal{L}$ is well defined, i.e. that $\overline{\mathcal{C}}L$ is an invertible matrix, and, secondly, that $[\mathcal{L}, \overline{\mathcal{B}}, \mathcal{B}]$ has full rank (otherwise the matrix S is not well defined). Assume that $\overline{\mathcal{C}}Lm = 0$ for some $m \in \mathbb{R}^n$. Then $Lm \in \operatorname{im} L \cap \ker\overline{\mathcal{C}} = \{0\}$, hence $\overline{\mathcal{C}}L$ has only a trivial kernel which implies invertibility. To show that $[\mathcal{L}, \overline{\mathcal{B}}, \mathcal{B}]$ has full rank, observe that $\operatorname{im}\mathcal{L} = \operatorname{im} L$ and, by the definition of the relative degree, $\operatorname{im}\left[\overline{\mathcal{B}}, \mathcal{B}\right] \subseteq \ker\overline{\mathcal{C}}$. Hence $\{0\} = \operatorname{im} L \cap \ker\overline{\mathcal{C}} \supseteq \operatorname{im}\mathcal{L} \cap \operatorname{im}\left[\overline{\mathcal{B}}, \mathcal{B}\right]$, which implies that $[\mathcal{L}, \overline{\mathcal{B}}, \mathcal{B}]$ has full rank.

Step 1d: The matrix $\overline{\mathcal{L}}$.
If $d_b = r+1$, then $\overline{\mathcal{L}}$ is the empty matrix. Otherwise choose, analogously as in the previous step, a full rank matrix $K \in \mathbb{R}^{(d_b-r-1)\times n}$ such that $\operatorname{im} K^\top \oplus \ker\left[\overline{\mathcal{B}}, \mathcal{B}\right]^\top = \ker\mathcal{B}^\top$. Again the matrix $\overline{\mathcal{B}}^\top K^\top$ is invertible. Let

$$\mathcal{K} = (K\overline{\mathcal{B}})^{-1}K,$$

with an analogous argument as in Step 1c it can be shown that $\left[\mathcal{K}/\overline{\mathcal{C}}/\mathcal{C}\right]$ has full rank, hence it is possible to choose a full rank matrix $\overline{\mathcal{L}} \in \mathbb{R}^{n\times(n-d_c-d_b+r+1)}$ such that

$$\operatorname{im}\overline{\mathcal{L}} = \ker\left[\mathcal{K}/\overline{\mathcal{C}}/\mathcal{C}\right].$$

It remains to show that $\left[\mathcal{L}, \overline{\mathcal{L}}, \overline{\mathcal{B}}, \mathcal{B}\right]$ has full rank (i.e. is invertible). To show this, first observe that, by the definition of the relative degree, $\operatorname{im}\mathcal{B} \cap \ker\mathcal{C} = \{0\}$ and recall that $\operatorname{im}\mathcal{L} \cap \ker\overline{\mathcal{C}} = \{0\}$ and analogously $\operatorname{im}\mathcal{K}^\top \cap \ker\overline{\mathcal{B}}^\top = \{0\}$, the latter is equivalent to $\operatorname{im}\overline{\mathcal{B}} \cap \ker\mathcal{K} = \{0\}$. Altogether this yields

$$\ker\left[\mathcal{K}/\overline{\mathcal{C}}/\mathcal{C}\right] \cap \operatorname{im}\left[\mathcal{L}, \overline{\mathcal{B}}, \mathcal{B}\right] = \{0\},$$

which implies that the square matrix $\left[\mathcal{L}, \overline{\mathcal{L}}, \overline{\mathcal{B}}, \mathcal{B}\right]$ has full rank which completes the first step of the proof.

Step 2.
Step 2a: $ST = \hat{I}$.
By definition $ST = \hat{I}$.
Step 2b: $Sb = \hat{b}$.
Let $e_r = [0, \ldots, 0, 1, \underbrace{0, \ldots, 0}_{r}]^\top \in \mathbb{R}^n$ then $Sb = \hat{b} = \gamma e_r$ if, and only if, $b = \gamma S^{-1} e_r$. The latter is fulfilled since

$$\gamma S^{-1} = \big[\mathcal{L}, \overline{\mathcal{L}}, \overline{\mathcal{B}}, \underbrace{b, Nb, \ldots, N^r b}_{=\mathcal{B}}\big].$$

Step 2c: $cT = \hat{c}$.
Choose a full rank matrix $\overline{\mathcal{K}} \in \mathbb{R}^{(n-d_c-d_b+r+1)\times n}$ such that

$$\operatorname{im} \overline{\mathcal{K}}^\top = \ker \left[\mathcal{L}, \overline{\mathcal{B}}, \mathcal{B}\right]^\top.$$

It can be shown analogously as in Step 1d that the square matrix $\mathfrak{C} := \left[\overline{\mathcal{C}}/\overline{\mathcal{K}}/\mathcal{K}/\mathcal{C}\right]$ has full rank (i.e. is invertible). Writing $\mathfrak{B} := \left[\mathcal{L}, \overline{\mathcal{L}}, \overline{\mathcal{B}}, \mathcal{B}\right]$ the matrix T can be written as

$$T = \mathfrak{C}^{-1} \gamma^{-1} \mathfrak{C}\mathfrak{B}\hat{I}.$$

Since $c\mathfrak{C}^{-1} = [0, \ldots, 0, 1]$ it remains to show that

$$[0, \ldots, 0, 1]\gamma^{-1}\mathfrak{C}\mathfrak{B}\hat{I} = [0, \ldots, 0, 1] = \hat{c},$$

or, equivalently, that the last row of the product $\mathfrak{C}\mathfrak{B}$ equals the last row of $\gamma \hat{I}^{-1}$. It is easy to see that the last row of $\gamma \hat{I}^{-1}$ is $[0, \ldots, 0, cb, cNb, \ldots, cN^r b]$. Observe that

$$\mathfrak{C}\mathfrak{B} = \begin{bmatrix} \overline{\mathcal{C}}\mathcal{L} & \overline{\mathcal{C}}\overline{\mathcal{L}} & \overline{\mathcal{C}}\overline{\mathcal{B}} & \overline{\mathcal{C}}\mathcal{B} \\ \overline{\mathcal{K}}\mathcal{L} & \overline{\mathcal{K}}\overline{\mathcal{L}} & \overline{\mathcal{K}}\overline{\mathcal{B}} & \overline{\mathcal{K}}\mathcal{B} \\ \mathcal{K}\mathcal{L} & \mathcal{K}\overline{\mathcal{L}} & \mathcal{K}\overline{\mathcal{B}} & \mathcal{K}\mathcal{B} \\ \mathcal{C}\mathcal{L} & \mathcal{C}\overline{\mathcal{L}} & \mathcal{C}\overline{\mathcal{B}} & \mathcal{C}\mathcal{B} \end{bmatrix}. \tag{4}$$

The matrices $\mathcal{L}$ and $\overline{\mathcal{L}}$ are such that $\operatorname{im} \mathcal{L}$ and $\operatorname{im} \overline{\mathcal{L}}$ are both subspaces of $\ker \mathcal{C}$, hence $\mathcal{C}\mathcal{L} = 0$ and $\mathcal{C}\overline{\mathcal{L}} = 0$. From the definition of the relative degree it follows that $\overline{\mathcal{C}}\mathcal{B} = 0$. Together with (3) this shows that the last row of $\mathfrak{C}\mathfrak{B}$

is $[0,\dots,0,cb,cNb,\dots,cN^rb]$.
Step 2d: SNT.
Invoking the notation of Step 2c write

$$SNT = (\mathfrak{C}\mathfrak{B})^{-1}\mathfrak{C}N\mathfrak{B}\hat{I}.$$

Note that the product $\mathfrak{C}\mathfrak{B}$ in (4) can further be simplified by the following observations, $\overline{\mathcal{C}}\mathcal{L} = \gamma I$, $\overline{\mathcal{C}}[\overline{\mathcal{L}},\overline{\mathcal{B}},\mathcal{B}] = 0$, $\overline{\mathcal{K}}[\mathcal{L},\overline{\mathcal{B}},\mathcal{B}] = 0$, $\mathcal{K}[\overline{\mathcal{L}},\mathcal{B}] = 0$, and $\mathcal{K}\overline{\mathcal{B}} = I$:

$$\mathfrak{C}\mathfrak{B} = \begin{bmatrix} \gamma I & 0 & 0 & 0 \\ 0 & \overline{\mathcal{K}\mathcal{L}} & 0 & 0 \\ \mathcal{K}\mathcal{L} & 0 & I & 0 \\ 0 & 0 & 0 & \mathcal{C}\mathcal{B} \end{bmatrix}.$$

Hence

$$(\mathfrak{C}\mathfrak{B})^{-1} = \begin{bmatrix} \gamma^{-1} I & 0 & 0 & 0 \\ 0 & (\overline{\mathcal{K}\mathcal{L}})^{-1} & 0 & 0 \\ -\gamma^{-1}\mathcal{K}\mathcal{L} & 0 & I & 0 \\ 0 & 0 & 0 & (\mathcal{C}\mathcal{B})^{-1} \end{bmatrix}.$$

By Proposition 16

$$N\overline{\mathcal{B}} = \overline{\mathcal{B}}\left[\begin{smallmatrix} 0 & & \\ 1 & \ddots & \\ & 1 & 0 \end{smallmatrix}\right] \quad \text{and} \quad \overline{\mathcal{C}}N = \left[\begin{smallmatrix} 0 & & \\ 1 & \ddots & \\ & 1 & 0 \end{smallmatrix}\right]\overline{\mathcal{C}},$$

furthermore $\mathcal{C}N\mathcal{B} = \mathcal{C}\mathcal{B}\left[\begin{smallmatrix} 0 & & \\ 1 & \ddots & \\ & 1 & 0 \end{smallmatrix}\right]$, $\overline{\mathcal{K}}N\mathcal{B} = 0$ and $\mathcal{C}N\overline{\mathcal{L}} = 0$, hence

$$\mathfrak{C}N\mathfrak{B} = \begin{bmatrix} \gamma\left[\begin{smallmatrix} 0 & & \\ 1 & \ddots & \\ & 1 & 0 \end{smallmatrix}\right] & 0 & 0 & 0 \\ \overline{\mathcal{K}}N\mathcal{L} & \overline{\mathcal{K}}N\overline{\mathcal{L}} & 0 & 0 \\ \mathcal{K}N\mathcal{L} & \mathcal{K}N\overline{\mathcal{L}} & \left[\begin{smallmatrix} 0 & & \\ 1 & \ddots & \\ & 1 & 0 \end{smallmatrix}\right] & \mathcal{K}N\mathcal{B} \\ \mathcal{C}N\mathcal{L} & 0 & 0 & \mathcal{C}\mathcal{B}\left[\begin{smallmatrix} 0 & & \\ 1 & \ddots & \\ & 1 & 0 \end{smallmatrix}\right] \end{bmatrix}.$$

Therefore,

$$SNT = (\mathfrak{C}\mathfrak{B})^{-1}\mathfrak{C}N\mathfrak{B}\hat{I} = \begin{bmatrix} \left[\begin{smallmatrix} 0 & & \\ 1 & \ddots & \\ & 1 & 0 \end{smallmatrix}\right] & 0 & 0 & 0 \\ \hat{E}_1 & \hat{N}_1 & 0 & 0 \\ E_2 & \hat{E}_3 & \left[\begin{smallmatrix} 0 & & \\ 1 & \ddots & \\ & 1 & 0 \end{smallmatrix}\right] & \mathcal{K}N\mathcal{B}I_* \\ (\mathcal{C}\mathcal{B})^{-1}\mathcal{C}N\mathcal{L} & 0 & 0 & \left[\begin{smallmatrix} 0 & & \\ 1 & \ddots & \\ & 1 & 0 \end{smallmatrix}\right] I_* \end{bmatrix},$$

where $\hat{E}_1 = (\overline{\mathcal{K}\mathcal{L}})^{-1}\overline{\mathcal{K}}N\mathcal{L}$, $\hat{N}_1 = (\overline{\mathcal{K}\mathcal{L}})^{-1}\overline{\mathcal{K}}N\overline{\mathcal{L}}$, $E_2 = -\mathcal{K}\mathcal{L}\left[\begin{smallmatrix}0&&\\1&\ddots&\\&\ddots&\ddots\\&&1\ 0\end{smallmatrix}\right] + \mathcal{K}N\mathcal{L}$, $\hat{E}_3 = \mathcal{K}N\overline{\mathcal{L}}$, and I_* is given by (2). Note that

$$\mathcal{C}N\mathcal{L} = \begin{bmatrix} cN^{r+1} \\ cN^r \\ \vdots \\ cN^2 \\ cN \end{bmatrix}\mathcal{L} = \begin{bmatrix} \gamma cN^{r+1}L(\overline{\mathcal{C}}L)^{-1} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} = \gamma \begin{bmatrix} 0 \text{———} 0\ 1 \\ 0 \\ \end{bmatrix},$$

hence $(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}N\mathcal{L} = 0_*$, and

$$\mathcal{K}N\mathcal{B} = \mathcal{K}[Nb, N^2b, \ldots, N^rb, N^{r+1}b] = [0, 0, \ldots, 0, (K\overline{\mathcal{B}})^{-1}KN^{r+1}b] = \begin{bmatrix} & & 1 \\ & & 0 \\ & 0 & \vert \\ & & \vert \\ & & 0 \end{bmatrix},$$

hence $\mathcal{K}N\mathcal{B}I_* = 0^*$. Clearly, $\left[\begin{smallmatrix}0&&\\1&\ddots&\\&\ddots&\ddots\\&&1\ 0\end{smallmatrix}\right] I_* = \left[\begin{smallmatrix}0&&\\1&\ddots&\\&\ddots&\ddots\\&&1\ 0\end{smallmatrix}\right]$ and it remains to show that $\hat{N}_1$ is nilpotent. This follows from the fact that $SNT\hat{I}^{-1} = SNS^{-1}$ is nilpotent and because of the special block structure this implies that $\hat{N}_1$ must also be nilpotent. Without changing the block structure it is possible to transform $\hat{N}_1$ to Jordan form N_1, this changes $\hat{E}_1$ and $\hat{E}_3$ to E_1 and E_3.

qed

Remark 21. *The proof of Theorem 20 is constructive. In fact, for a given DAS in standard form $(c, N, I, b) \in \Sigma_n^{\text{pure}}$ with negative relative degree $r \geq 0$ and impulse-controllability- and impulse-observability-indices $d_b, d_c \in \mathbb{N}$ the* *specific matrices in the normal form are given as follows:*

$$E_1 = J^{-1}(\overline{\mathcal{K}}\overline{\mathcal{L}})^{-1}\overline{\mathcal{K}}N\mathcal{L} \in \mathbb{R}^{(n-d_c-d_b+r+1)\times(d_b-r-1)},$$

$$E_2 = \mathcal{K}N\mathcal{L} - \mathcal{K}\mathcal{L}\begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & 1 & 0 \end{bmatrix} \in \mathbb{R}^{(d_b-r-1)\times(d_c-r-1)},$$

$$E_3 = \mathcal{K}N\overline{\mathcal{L}}J \in \mathbb{R}^{(d_b-r-1)\times(n-d_c-d_b+r+1)},$$

$$N_1 = J^{-1}(\overline{\mathcal{K}}\overline{\mathcal{L}})^{-1}\overline{\mathcal{K}}N\overline{\mathcal{L}}J \in \mathbb{R}^{(n-d_c-d_b+r+1)\times(n-d_c-d_b+r+1)},$$

$$0^* = \begin{bmatrix} \frac{cb}{-\gamma} & \frac{cNb}{-\gamma} & \text{—} & \frac{cN^{r-1}b}{-\gamma} & 1 \\ & & 0 & & \end{bmatrix} \in \mathbb{R}^{(d_b-r-1)\times(r+1)},$$

$$0_* = (\mathcal{C}\mathcal{B})^{-1}\gamma \begin{bmatrix} & 1 \\ & 0 \\ 0 & | \\ & 0 \end{bmatrix} \in \mathbb{R}^{(r+1)\times(d_c-r-1)},$$

$$I_* = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & & & \\ \frac{cb}{-\gamma} & \frac{cNb}{-\gamma} & \text{—} & \frac{cN^{r-1}b}{-\gamma} & 1 \end{bmatrix} \in \mathbb{R}^{(r+1)\times(r+1)},$$

where

$$\mathcal{B} := [b, Nb, \ldots, N^r b], \quad \overline{\mathcal{B}} := [N^{r+1}b, N^{r+2}b, \ldots, N^{d_b-1}b]$$

$$\overline{\mathcal{C}} := [cN^{d_c-1}/cN^{d_c-2}/\cdots/cN^{r+1}], \quad \mathcal{C} := [cN^r/\ldots/cN/c],$$

$$\mathcal{K} := [0, I]\left([\mathcal{B}^\top/\overline{\mathcal{B}}^\top][\mathcal{B},\overline{\mathcal{B}}]\right)^{-1}\left[\mathcal{B}^\top/\overline{\mathcal{B}}^\top\right] \in \mathbb{R}^{(d_b-r-1)\times n}$$

$$\mathcal{L} := \gamma[\overline{\mathcal{C}}^\top, \mathcal{C}^\top]\left([\overline{\mathcal{C}}/\mathcal{C}][\overline{\mathcal{C}}^\top, \mathcal{C}^\top]\right)^{-1}[I/0] \in \mathbb{R}^{n\times(d_c-r-1)},$$

$$\overline{\mathcal{K}}^\top \in \mathbb{R}^{n\times(n-d_c-d_b+r+1)} \text{ is a basis of } \ker\left[\mathcal{L}^\top/\overline{\mathcal{B}}^\top/\mathcal{B}^\top\right],$$

$$\overline{\mathcal{L}} \in \mathbb{R}^{n\times(n-d_c-d_b+r+1)} \text{ is a basis of } \ker\left[\mathcal{K}/\overline{\mathcal{C}}/\mathcal{C}\right],$$

and $J \in \mathbb{R}^{(n-d_c-d_b+r+1)\times(n-d_c-d_b+r+1)}$ *is a basis transformation such that* N_1 *is in Jordan normal form.*
If the DAS (c, E, A, b) *is not in standard form, then either* N *and* b *in the above formulae must be replaced by* $A^{-1}E$ *and* $A^{-1}b$*, resp., or* N *and* c *must be replaced by* EA^{-1} *and* cA^{-1}*, resp.*

Remark 22. *If the negative relative degree of a pure DAE is maximal, i.e.* $r = n-1$*, then the normal form above coincides with the minimal realization in R-form as given in Proposition 11.*

Corollary 23. *All minimal realizations of a pure DAS are equivalent.*

The normal form of Theorem 20 can be viewed as a specialization of (2-5.4) in [3, p. 52]: it is more explicit and simpler, the size of the different blocks is explicitly given, and the influence of the input on the states can be seen more directly as well as the influence of the states on the output. Furthermore no proof is given in [3].

6. Impulse-controllability and -observability revisited

With the normal form from Theorem 20 it is now possible to give characterization of impulse-controllability and -observability.

Theorem 24. *Consider $(c, E, A, b) \in \Sigma_n^{\text{pure}}$, $n \in \mathbb{N}$, with negative relative degree $r \geq 0$, impulse-controllability- and impulse-observability-indices $d_b, d_c \in \mathbb{N}$ and let $N_1 \in \mathbb{R}^{(n-d_b-d_c+r+1)\times(n-d_b-d_c+r+1)}$ be given as in Theorem 20. Then the following characterizations of impulse-controllability and -observability hold:*

(i) *The DAS is impulse-controllable if, and only if, $d_c = r + 1$ and $N_1 = 0$*

(ii) *The DAS is impulse-observable if, and only if, $d_b = r + 1$ and $N_1 = 0$.*

Proof. Let $(\hat{c}, \hat{N}, \hat{I}, \hat{b})$ be the normal form of (c, E, A, b) from Theorem 20.

(i) It is easily seen that $(\hat{c}, \hat{N}, \hat{I}, \hat{b})$ is equivalent to $(\tilde{c}, \tilde{N}, I, \tilde{b})$ with

$$\tilde{N} = \begin{bmatrix} \begin{smallmatrix}0&&\\1&\ddots&\\&1&0\end{smallmatrix} & 0 & 0 & 0 \\ E_1 & N_1 & 0 & 0 \\ 0_* & 0 & \begin{smallmatrix}0&&\\1&\ddots&\\&1&0\end{smallmatrix} & 0 \\ E_2 & E_3 & \widetilde{0^*} & \begin{smallmatrix}0&&\\1&\ddots&\\&1&0\end{smallmatrix} \end{bmatrix}$$

where the matrices $E_1, E_2, E_3, N_1, 0_*$ are the same as in Theorem 20 and $\widetilde{0^*}$ has the same structure as 0^* from Theorem 20, in particular 0_* and $\widetilde{0^*}$ have a one in the upper right corner. The vector $\tilde{b}$ is given by $\tilde{b} = [0, \ldots, 0, \gamma, 0, \ldots, 0]^\top$ with $\gamma \neq 0$ at the $(n - d_b + 1)$-th position. It is easily seen that

$$\operatorname{im}[\tilde{N}\tilde{b}, \tilde{N}^2\tilde{b}, \ldots, \tilde{N}^{n-1}\tilde{b}] = \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \begin{smallmatrix}0&&\\1&\ddots&\\&1&0\end{smallmatrix} & 0 \\ \widetilde{0^*} & \begin{smallmatrix}0&&\\1&\ddots&\\&1&0\end{smallmatrix} \end{bmatrix},$$

which implies that impulse-controllability for the given DAS is equivalent to the condition

$$\operatorname{im}\begin{bmatrix} \begin{smallmatrix}0&&\\1&\ddots&\\&1&0\end{smallmatrix} & 0 \\ E_1 & N_1 \\ 0_* & 0 \\ E_2 & E_3 \end{bmatrix} \subseteq \operatorname{im}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \begin{smallmatrix}0&&\\1&\ddots&\\&1&0\end{smallmatrix} & 0 \\ \widetilde{0^*} & \begin{smallmatrix}0&&\\1&\ddots&\\&1&0\end{smallmatrix} \end{bmatrix}.$$

A necessary and sufficient condition for this is that the matrix 0_* is not existent (because it has a one in the upper right corner), i.e. $d_c = r+1$, and that N_1 is the zero matrix.

(ii) It is easily seen that $(\hat{c}, \hat{N}, \hat{I}, \hat{c})$ is equivalent to $(\tilde{c}, \tilde{N}, I, \tilde{b})$ with

$$\tilde{N} = \begin{bmatrix} \begin{smallmatrix}0&&\\1&\ddots&\\&1&0\end{smallmatrix} & 0 & 0 & 0 \\ \widetilde{0_*} & \widetilde{\left[\begin{smallmatrix}0&&\\1&\ddots&\\&1&0\end{smallmatrix}\right]} & 0 & 0 \\ E_1 & 0 & N_1 & 0 \\ E_2 & 0^* & E_3 & \begin{smallmatrix}0&&\\1&\ddots&\\&1&0\end{smallmatrix} \end{bmatrix},$$

where E_1, E_2, E_3, N_1, O^* are as in the normal form in Theorem 20, $\widetilde{0_*}$ has the same structure as 0_* from Theorem 20 and

$$\widetilde{\left[\begin{smallmatrix}0&&\\1&\ddots&\\&1&0\end{smallmatrix}\right]} = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & 1 & 0 & \\ * & \cdots & * & 1 & 0 \end{bmatrix}.$$

The vector $\tilde{c}$ is given by $\tilde{c} = [0, \ldots, 0, 1, 0, \ldots, 0]$ where the one is at position d_c. Easy calculations show that (here it is needed that $\widetilde{0_*}$ has a one in the upper right corner)

$$\ker[\tilde{c}\tilde{N}^{d_c}/\tilde{c}\tilde{N}^{d_c-1}/\ldots/\tilde{c}\tilde{N}] = \ker\begin{bmatrix} \begin{smallmatrix}0&&\\1&\ddots&\\&1&0\end{smallmatrix} & 0 & 0 & 0 \\ \widetilde{0_*} & \widetilde{\left[\begin{smallmatrix}0&&\\1&\ddots&\\&1&0\end{smallmatrix}\right]} & 0 & 0 \end{bmatrix}.$$

Hence impulse-controllability is equivalent to the condition

$$\ker\begin{bmatrix} \begin{smallmatrix}0&&\\1&\ddots&\\&1&0\end{smallmatrix} & 0 & 0 & 0 \\ \widetilde{0_*} & \widetilde{\left[\begin{smallmatrix}0&&\\1&\ddots&\\&1&0\end{smallmatrix}\right]} & 0 & 0 \end{bmatrix} \subseteq \ker\begin{bmatrix} E_1 & 0 & N_1 & 0 \\ E_2 & 0^* & E_3 & \begin{smallmatrix}0&&\\1&\ddots&\\&1&0\end{smallmatrix} \end{bmatrix}.$$

Because 0^* has a one in the upper right corner the inclusion holds if, and only if, 0^* does not exists, i.e. $d_b = r + 1$, and $N_1 = 0$.

$\boxed{\text{qed}}$

Corollary 25. *A pure DAS with state space dimension $n \in \mathbb{N}$ is impulse-controllable and -observable if, and only if, it is equivalent to*

$$(c, N, \hat{I}, b) = \left([0, \ldots, 0, 1], \begin{bmatrix} 0 & 0 \\ 0 & \begin{smallmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & \ddots \\ & & 1 \; 0 \end{smallmatrix} \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & \begin{smallmatrix} 1 & & \\ & \ddots & \\ * & \cdots\; * & 1 \end{smallmatrix} \end{bmatrix} [\underbrace{0, \ldots, 0}_{n-r-1}, \gamma, \underbrace{0, \ldots, 0}_{r}]^\top \right),$$

where the diagonal square blocks of N and $\hat{I}$ have size $(n - r - 1)$ and $(r + 1)$ for some $r \in \{0, \ldots, n - 1\}$ and $\gamma \neq 0$.

Corollary 26. (i) *Two pure DASs with the same state space dimension which are impulse-controllable and -observable are equivalent if, and only if, they are IO-equivalent.*

(ii) *If the negative relative degree of a pure DAS $(c, E, A, b) \in \Sigma_n^{\text{pure}}$, $n \in \mathbb{N}$, is maximal, i.e. $r = n - 1$, then (c, E, A, b) is impulse-controllable and -observable.*

(iii) *All minimal realization of a pure DAS are impulse-controllable and -observable.*

Remark 27. *Note that an impulse-controllable and -observable DAS which is a realization of a polynomial transfer function need not to be minimal, because one can add arbitrarily many "trivial" state equations $z_1 = 0$, $z_2 = 0$, $\ldots$, $z_N = 0$ without loosing the property of impulse- controllability and -observability.*

7. Conclusion

For pure differential algebraic equation a normal form is derived which shows very clearly the influence of the input on the states, the influence of the states on the output and the relative degree. The normal form also separates the states into impulse-controllable and -observable states and easy characterizations of impulse-controllability and -observability based on the normal form are given. Some specific minimal realizations of pure DAEs are given and connections between the relative degree and the normal form are highlighted.

In combination with distributional solution theory the normal form might be used in future research to study the influence of inconsistent initial values on the output and the influence of non-smooth inputs on the states and the output. The normal form might also help for synthesis of controllers for specific control tasks, e.g. impulse elimination.
Finally the proof of the normal is constructive, i.e. it is possible to calculate the transformation matrices and the normal form explicitly, nevertheless the given formulae are not studied with respect to numerical feasibility.

Acknowledgment

I thank Achim Ilchmann who gave valuable hints and suggestions which helped to improve the clarity and quality of the paper. I would also like to thank Vu Hoang Linh (Hanoi Univ., Vietnam) whose visit in Ilmenau initiated the work on this paper.

References

[1] S. L. Campbell, Singular Systems of Differential Equations I, Pitman, New York, 1980.

[2] S. L. Campbell, Singular Systems of Differential Equations II, Pitman, New York, 1982.

[3] L. Dai, Singular Control Systems, no. 118 in Lecture Notes in Control and Information Sciences, Springer-Verlag, Berlin, 1989.

[4] P. J. Rabier, W. C. Rheinboldt, Theoretical and Numerical Analysis of Differential-Algebraic Equations, Vol. VIII of Handbook of Numerical Analysis, Elsevier Publications, Amsterdam, The Netherlands, 2002.

[5] P. Kunkel, V. Mehrmann, Differential-Algebraic Equations. Analysis and Numerical Solution, EMS Publishing House, Zürich, Switzerland, 2006.

[6] A. Isidori, Nonlinear Control Systems, 3rd Edition, Communication and Control Engineering Series, Springer-Verlag, Berlin, 1995.

[7] A. Ilchmann, E. Ryan, P. Townsend, Tracking with prescribed transient behaviour for nonlinear systems of known relative degree, SIAM J. Control Optimization 46 (1) (2007) 210–230.

[8] R. Kalman, Canonical structure of linear dynamical systems, Proc. Nat. Acad. Sci. U.S.A. 48 (4) (1962) 596–600.

[9] G. C. Verghese, B. C. Lévy, T. Kailath, A generalized state-space for singular systems, IEEE Trans. Automat. Contr. AC-26 (4) (1981) 811–831.

[10] J. Loiseau, K. Özçaldiran, M. Malabre, N. Karcanias, Feedback canonical forms of singular systems, Kybernetika 27 (4) (1991) 289–305.

[11] W. Rath, Derivative and proportional state feedback for linear descriptor systems with variable coefficients, Lin. Alg. Appl. 260 (1997) 273–310.

[12] K. Weierstraß, Zur Theorie der bilinearen und quadratischen Formen, Monatsh. Akad. Wiss. (1867) 310–338.

[13] D. J. Cobb, Controllability, observability and duality in singular systems, IEEE Trans. Automat. Contr. AC-29 (1984) 1076–1082.

[14] S. Lang, Linear Algebra, 2nd Edition, Addison-Wesley, Reading, Massachusetts, 1970, fourth printing 1973.