A converse Lyapunov theorem for switched DAEs

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For switched ordinary differential equations (ODEs) it is well known that exponential stability under arbitrary switching yields the existence of a common Lyapunov function. The result is known as a "converse Lyapunov Theorem". In this note we will present a converse Lyapunov theorem for switched differential algebraic equations (DAEs) as well as the construction of a Barabanov norm for irreducible switched DAEs.

1 Introduction

We consider linear switched DAEs (differential algebraic equations) of the form

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) \quad \text{or short} \quad E_\sigma \dot{x} = A_\sigma x, \tag{1}$$

where $\sigma : \mathbb{R} \to \{1, 2, \dots, \mathrm{p}\}$ is a piecewise constant, right-continuous switching signal with locally finitely many jumps (no Zeno behavior). The matrix pairs $(E_1, A_1), \dots, (E_\mathrm{p}, A_\mathrm{p})$ with $E_p, A_p \in \mathbb{R}^{n\times n}$, $p = 1, \dots, \mathrm{p}$, are assumed to be regular, i.e. $\det(E_p s - A_p) \not\equiv 0$. For the purpose of this note we have to exclude impulsive solutions (but jumps in x will not be excluded), this can be guaranteed [2, Thm 3.8] by assuming that

$$\forall p, q \in \{1, 2, \dots, \mathrm{p}\} : E_q(I - \Pi_q)\Pi_p = 0,$$

where Π_p and Π_q denote the consistency projectors corresponding to the matrix pairs (E_p, A_p) and (E_q, A_q), respectively, cf. Theorem 2.1. We are interested in the following question:

$$E_\sigma \dot{x} = A_\sigma x \text{ uniformly asymptotically stable } \forall\sigma \quad \overset{?}{\Rightarrow} \quad \exists \text{ common Lyapunov function.}$$

It is well known that this question can be answered in the affirmative for switched ODEs (i.e. where $E_p = I$ for all p). In fact, one possible avenue of studying this problem is by means of Lyapunov norms:

Definition 1.1 (Lyapunov norm) $\|\!|\cdot|\!\|$ is called a λ-*Lyapunov norm*, $\lambda \in \mathbb{R}$, for the switched DAE (1) if, and only if, for all switching signals σ and all solutions x of (1)

$$\|\!|x(t)|\!\| \leq e^{\lambda t}\|\!|x(0-)|\!\|.$$

In particular, if we are able to find a λ-Lyapunov norm for $\lambda < 0$ then $V = \|\!|\cdot|\!\|$ defines a Lyapunov function for the switched DAE (1).

Due to space limitations we omit the proofs of our results, these and also a literature review will be published elsewhere.

2 Evolution operator and its semigroup

For a characterization of the solutions of the switched DAE (1) the following result is crucial.

Theorem 2.1 (A^{diff} and $\Pi_{(E,A)}$, [4]) *Let* $(E, A) \in (\mathbb{R}^{n\times s})^2$ *be regular and consider the DAE*

$$E\dot{x} = Ax \quad \textit{on } [0, \infty).$$

Then there exists an unique consistency projector $\Pi_{(E,A)} \in \mathbb{R}^{n\times n}$ *and an unique flow matrix* $A^{\text{diff}} \in \mathbb{R}^{n\times n}$ *such that for all solutions* x *of the above DAE it holds that*

$$x(0) = \Pi_{(E,A)}x(0-)$$

and

$$\dot{x} = A^{\text{diff}}x \quad \textit{on } (0, \infty).$$

Furthermore, $A^{\text{diff}}\Pi_{(E,A)} = \Pi_{(E,A)}A^{\text{diff}}$.

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A direct consequence of this result and the impulse-freeness assumption is the following solution formula.

Corollary 2.2 (Solution formula for switched DAE) *Any solution of the switched DAE* $E_\sigma \dot{x} = A_\sigma x$ *has the form*

$$x(t) = e^{A_k^{\text{diff}}(t-t_k)}\Pi_k e^{A_{k-1}^{\text{diff}}(t_k-t_{k-1})}\Pi_{k-1}\cdots e^{A_1^{\text{diff}}(t_2-t_1)}\Pi_1 e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0 x(t_0-),$$

where $t_0 < t_1 < t_2 < \ldots$ *are the switching times of* σ*,* k *is such that* $t \in [t_k, t_{k+1})$ *and* A_i^{diff}, Π_i*,* $i = 0, 1, \ldots, k$*, are the flow matrices and consistency projectors corresponding the matrix pair* $(E_{\sigma(t_i)}, A_{\sigma(t_i)})$ *active on the interval* $[t_i, t_{i+1})$*.*

Using the same notation as in the previous corollary, we define the evolution operator corresponding to the switched DAE (1) as follows, for all $t_0, t \in \mathbb{R}$ with $t \geq t_0$,

$$\Phi^\sigma(t, t_0) := e^{A_k^{\text{diff}}(t-t_k)}\Pi_k e^{A_{k-1}^{\text{diff}}(t_k-t_{k-1})}\Pi_{k-1}\cdots e^{A_1^{\text{diff}}(t_2-t_1)}\Pi_1 e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0 \in \mathbb{R}^{n\times n}.$$

Then any solution x of the switched DAE (1) fulfills, for all $t \geq t_0 \in \mathbb{R}$,

$$x(t) = \Phi^\sigma(t, t_0)x(t_0-).$$

Since we consider arbitrary switching signals we can define the set of all evolution operators covering a certain time span Δt.

Definition 2.3 (Set of all evolutions with fixed time span $\Delta t \geq 0$)

$$\begin{aligned}\mathcal{S}_{\Delta t} &:= \bigcup \left\{\ \Phi^\sigma(t_0 + \Delta t, t_0)\ \middle|\ t_0 \in \mathbb{R},\ \text{piecewise-constant switching signals } \sigma\ \right\}\\ &= \left\{\ \prod_{i=0}^{k} e^{A_i^{\text{diff}}\tau_i}\Pi_i\ \middle|\ (A_i^{\text{diff}}, \Pi_i) \in \mathcal{M},\ \sum_{i=0}^{k}\tau_i = \Delta t,\ \tau_i > 0, i = 0, \ldots, k-1, \tau_k \geq 0\ \right\} \subseteq \mathbb{R}^{n\times n},\end{aligned}$$

where $\mathcal{M} := \left\{\ (A_p^{\text{diff}}, \Pi_p)\ \middle|\ \text{corresponding to } (E_p, A_p), p = 1, \ldots, \mathtt{p}\ \right\}$.

Note that the following equivalence holds

$$x \text{ solves } E_\sigma\dot{x} = A_\sigma x \text{ for some } \sigma \quad \Leftrightarrow \quad \forall t_0 \in \mathbb{R}\ \forall \Delta t \geq 0\ \exists \Phi_{\Delta t} \in \mathcal{S}_{\Delta t}: \quad x(t_0 + \Delta t) = \Phi_{\Delta t}x(t_0-).$$

With the help of the sets $\mathcal{S}_t$, $t \geq 0$, we can now define the exponential growth bound of the switched DAE (1).

Definition 2.4 (Exponential growth bound) For $t > 0$ the *exponential growth bound* of the switched DAE (1) is defined as

$$\lambda_t(\mathcal{S}_t) := \sup_{\Phi_t \in \mathcal{S}_t} \frac{\ln \|\Phi_t\|}{t} \in \mathbb{R} \cup \{-\infty, \infty\},$$

where $\|\cdot\|$ is the induced matrix norm for an arbitrary (but fixed in the following) norm $\|\cdot\|$ on $\mathbb{R}^n$.

This definition implies for all solutions x of $E_\sigma\dot{x} = A_\sigma x$:

$$\|x(t)\| = \|\Phi_t x(0-)\| \leq \|\Phi_t\|\,\|x(0-)\| \leq e^{\lambda_t(\mathcal{S}_t)\,t}\|x(0-)\|.$$

In contrast to switched ODEs without jumps where $\lambda_t(\mathcal{S}_t)$ is always finite, the cases $\lambda_t(\mathcal{S}_t) = -\infty$ and $\lambda_t(\mathcal{S}_t) = \infty$ are both possible! While the case $\lambda_t(\mathcal{S}_t) = -\infty$ occurs only exceptionally when all consistency projectors are trivial, i.e. $\Pi_p = 0$; the case $\lambda_t(\mathcal{S}_t) = \infty$ is more relevant as the following example shows.

Example 2.5 Consider a switched DAE (1) with $\mathtt{p} = 2$ modes and the matrix pairs (the colors correspond to the ones used in Figure 1)

$$(E_1, A_1) = \left(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & -1\\0 & -1\end{bmatrix}\right), \quad (E_2, A_2) = \left(\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}, \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}\right).$$

One possible solution of the switched DAE is shown in the left part of Figure 1. The corresponding evolution of the norm of the solution is shown in the middle of Figure 1. Increasing the switching frequency yields a faster growing of the norm of the state (see right part of Figure 1), in particular

$$\Phi_t \approx (\Pi_1\Pi_2)^k = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}^k = 2^{k-1}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

where k is the number of switches within the time span t. From this it becomes clear that the set $\mathcal{S}_t$ cannot be bounded, hence $\lambda_t(\mathcal{S}_t) = \infty$ for all $t > 0$.

Theorem 2.6 (Boundedness of $\mathcal{S}_t$) *Consider the switched DAE* (1) *with the associated set* $\mathcal{S}_t$ *of evolution operators for some* $t \geq 0$*. Then* $\mathcal{S}_t$ *is bounded if, and only if, the set of consistency projectors is product bounded, i.e. the set of all finite products of the consistency projectors is bounded.*

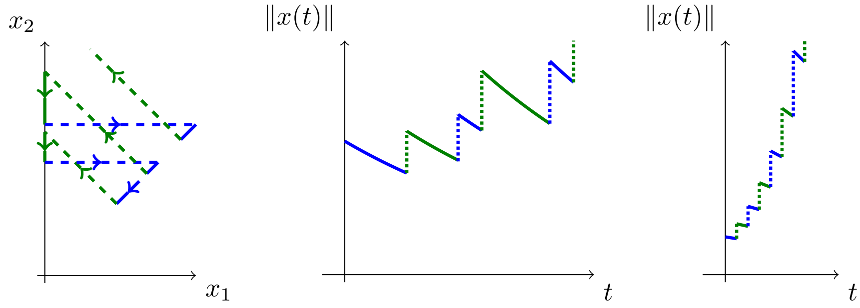

Fig. 1 One possible solution of the switched DAE from Example 2.5 (left) and the corresponding norm of the state (middle). If the switching frequency is increased the norm will grow faster (right).

The proof uses the fact (see e.g. [1, Thm. 3]) that product boundedness of the consistency projectors ensures existence of a norm $\|\cdot\|$ on $\mathbb{R}^n$ such that for the induced matrix norm it holds that $\|\Pi_{(E,A)}\| = 1$ for all consistency projectors.

Excluding the cases where $\lambda_t(\mathcal{S}_t)$ is not finite, we can now look at the asymptotical growth rate of the switched DAE (1). Therefore we first highlight an important property of the set of all evolution operators.

Lemma 2.7 (Semigroup) *The set*

$$\mathcal{S} := \bigcup_{t\geq 0} \mathcal{S}_t$$

is a semigroup with

$$\mathcal{S}_{s+t} = \mathcal{S}_s\mathcal{S}_t := \{ \Phi_s\Phi_t \mid \Phi_s \in \mathcal{S}_s, \Phi_t \in \mathcal{S}_t \}$$

In Theorem 2.1 we highlighted commutativity of the flow matrix A^{diff} with the consistency projector $\Pi_{(E,A)}$ for arbitrary regular matrix pairs (E, A). This property is needed to show the above equality, where the following intermediate step is essential:

$$e^{A^{\text{diff}}\tau}\Pi = e^{A^{\text{diff}}(\tau-\tau')}e^{A^{\text{diff}}\tau'}\Pi\Pi = e^{A^{\text{diff}}(\tau-\tau')}\Pi e^{A^{\text{diff}}\tau'}\Pi$$

for any $(A^{\text{diff}}, \Pi) \in \mathcal{M}$ and $0 \leq \tau' \leq \tau$.

We are now ready the formulate the first main result of this note.

Theorem 2.8 (Exponential growth rate well defined) *Let the consistency projectors be product bounded and not all be trivial, then the (upper) Lyapunov exponent*

$$\lambda(\mathcal{S}) := \lim_{t\to\infty} \lambda_t(\mathcal{S}_t) = \lim_{t\to\infty} \sup_{\Phi_t\in\mathcal{S}_t} \frac{\|\Phi_t\|}{t}$$

of the switched DAE (1) *is well defined and finite.*

The key idea to prove this result is to observe that submultiplicativity of induced matrix norms together with the semigroup property shown in Lemma 2.7 implies

$$(s + t)\lambda_{s+t}(\mathcal{S}_{s+t}) \leq s\lambda_s(\mathcal{S}_s) + t\lambda_t(\mathcal{S}_t).$$

3 Converse Lyapunov theorem and Barabanov norm

With the help of the Lyapunov exponent it is now possible to define a corresponding Lyapunov norm. However, if irreducibility is not assumed (cf. Theorem 3.4), then we can only find a λ-Lyapunov norm with $\lambda > \lambda(\mathcal{S})$, but λ can get arbitrarily close to the Lyapunov exponent $\lambda(\mathcal{S})$.

Theorem 3.1 (Lyapunov norm) *Consider the switched DAE* (1) *and assume its associated Lyapunov exponent* $\lambda(\mathcal{S})$ *is bounded. Then for each* $\varepsilon > 0$

$$\|x\|_\varepsilon := \sup_{t>0} \sup_{\Phi_t\in\mathcal{S}_t} e^{-(\lambda(\mathcal{S})+\varepsilon)t}\|\Phi_t x\|$$

defines a $(\lambda(\mathcal{S}) + \varepsilon)$*-Lyapunov norm for the switched DAE* (1).

The key argument to show well-definedeness is that by definition

$$\sup_{\Phi_t \in \mathcal{S}_t} \|\Phi_t x\| \leq e^{\lambda_t(\mathcal{S}_t)\,t} \|x\|$$

and hence, invoking Theorem 2.8,

$$e^{-(\lambda(\mathcal{S})+\varepsilon)t} \sup_{\Phi_t \in \mathcal{S}_t} \|\Phi_t x\| \to 0 \text{ as } t \to \infty.$$

We call the switched DAE (1) uniformly exponentially stable, when there exists $\lambda > 0$ and $M \geq 1$ such that for all switching signals, for all solutions x and all $t \geq t_0$:

$$\|x(t)\| \leq M e^{-\lambda t} \|x(t_0-)\|.$$

Clearly, in that case, $\lambda(\mathcal{S}) \leq -\lambda < 0$, hence we arrive at the following converse Lyapunov theorem.

Corollary 3.2 (Converse Lyapunov Theorem) *If the switched DAE* (1) *is uniformly exponentially stable then* $V = |||\cdot|||_\varepsilon$ *as in Theorem 3.1 for all* $\varepsilon > 0$ *sufficiently small is a Lyapunov function. In particular,* $V(\Pi x) \leq V(x)$ *for all consistency projectors* Π.

Remark 3.3 The norm $|||\cdot|||_\varepsilon$ is in general non-smooth. A smoothing procedure as in [3] might violate the jump condition $V(\Pi x) \leq V(x)$, hence it is not clear whether a smooth Lyapunov function can be found in general.

We finish this note by seeking a Lyapunov norm with a certain minimality property and which we call Barabanov norm.

Definition 3.4 (Barabanov norm) A norm $|||\cdot|||$ on $\mathbb{R}^n$ is called Barabanov norm for the switched DAE (1) if, and only if, there exists $\lambda \in \mathbb{R}$ with

1. $|||x(t)||| = |||\Phi_t x(0-)||| \leq e^{\lambda t} |||x(0-)|||, \quad \Phi_t \in \mathcal{S}_t$
2. $\forall x^0 \in \mathbb{R}^n\ \exists \overline{\Phi}_t \in \overline{\mathcal{S}_t}: \quad |||\overline{\Phi}_t x_0||| = e^{\lambda t} |||x_0|||,$ where $\overline{\mathcal{S}}_t$ denotes the closure of the set $\mathcal{S}_t$.

In general it is not possible to find a Barabanov norm because already in the unswitched case, the presence of non-trivial Jordan blocks yield polynomial solutions for which condition 2 cannot be satisfied. Also the presence of different eigenvalues in the unswitched case makes it impossible to satisfy condition 2 for all initial values $x \in \mathbb{R}^n$. However, if the set of all evolution operators $\mathcal{S}$ is irreducible, i.e. there are no trivial $\mathcal{S}$-invariant subspace $\mathcal{M} \subseteq \mathbb{R}^n$, or, in other words $\mathcal{S}\mathcal{M} \subseteq \mathcal{M}$ implies $\mathcal{M} = \emptyset$ or $\mathcal{M} = \mathbb{R}^n$, then we are able to construct a Barabanov norm.

Theorem 3.5 (Existence of Barabanov norm) *Consider the switched DAE* (1) *with the corresponding set* $\mathcal{S}$ *of evolution operators. Assume* $\mathcal{S}$ *is irreducible, then the following statements are equivalent:*

1. *The consistency projectors are product bounded.*
2. *The Lyapunov exponent* $\lambda(\mathcal{S})$ *is bounded.*
3. *There exists a Barabanov norm with* $\lambda = \lambda(\mathcal{S})$.

The construction of the Barabanov norm is similar as the construction in [5], we first observe that

$$\mathcal{S}_\infty := \bigcap_{T \geq 0} \overline{\bigcup_{t \geq T} e^{-\lambda(S)\,t} \mathcal{S}_t}$$

is a compact nontrivial irreducible semigroup. Now

$$|||x||| := \max\{\ \|Sx\|\ \mid S \in \mathcal{S}_\infty\ \}$$

is the desired Barabanov norm.

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