A preliminary result on synchronization of heterogeneous agents via funnel control

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***Abstract*— We propose a new approach to achieve practical synchronization for heterogeneous agents. Our approach is based on the observation that a sufficiently large (but constant) gain for diffusive coupling leads to practical synchronization. In the classical setup of high-gain adaptive control, the funnel controller gained popularity in the last decade, because it is very simple and only structural knowledge of the underlying dynamical system is needed. We illustrate with simulations that "funnel synchronization" may be a promising approach to achieve practical synchronization of heterogeneous agents without the need to know the individual dynamics and the algebraic connectivity of the network (i.e., the second smallest eigenvalue of the Laplacian matrix). For a special case we provide a proof, but the proof for the general case is ongoing research.**

I. Introduction

We consider heterogeneous agents, whose dynamics are given by

$$\dot{x}_i = f_i(t, x_i) + u_i, \quad x_i(0) = x_i^0 \quad i = 1, 2, \ldots, N, \tag{1}$$

where $N \in \mathbb{N}$ is the total number of agents, $x_i(t) \in \mathbb{R}$ is the (scalar) state-variable of agent i governed by the time-varying nonlinear vector field $f_i : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ and the input $u_i(t) \in \mathbb{R}$. We assume that the agents are coupled with each other via a graph (G, E) with the node set $G = \{1, 2, \ldots, N\}$ and the edge set $E \subseteq G \times G$, where $(j, i) \in E$ if, and only if, there is an information flow from agent j to agent i. We denote the adjacency matrix of (G, E) by $\mathcal{A} \in \{0, 1\}^{N\times N}$, i.e. $\mathcal{A}_{ij} = 1$ if and only if $(j, i) \in E$. Furthermore, let $1_N := (1, 1, \ldots, 1)^\top \in \mathbb{R}^N$ and

$$d = (d_1, d_2, \ldots, d_N)^\top := \mathcal{A}\,1_N, \qquad \mathcal{D} := \operatorname{diag}(d),$$
$$\mathcal{L} := \mathcal{D} - \mathcal{A}, \qquad \overline{\mathcal{L}} := \mathcal{D}^{-1}\mathcal{L},$$

in which d_i is the (in)-degree of the i-th node and $\mathcal{L}$ is the *Laplacian* of (G, E).

Suppose now that those agents are coupled by a proportional error feedback:

$$u_i = -k_i e_i, \quad i \in G, \tag{2}$$

where $k_i > 0$ is the coupling strength and the error e_i is the difference between the i-th agent's own state and the average of its neighbours' states, i.e.

$$e_i := x_i - \overline{x}_i := x_i - \frac{1}{d_i}\sum_{j\in\mathcal{N}_i} x_j, \tag{3}$$

where $\mathcal{N}_i := \{\ j \neq i \mid (j, i) \in E\ \}$ is the set of neighbors of agent i; in particular

$$e = \overline{\mathcal{L}}x.$$

Then the dynamics of the coupled agents can now be rewritten as

$$\dot{x} = f(t, x) - k\overline{\mathcal{L}}x, \quad x(0) = x^0,$$

where

$$f(t, x) := (f_1(t, x_1), \ldots, f_N(t, x_N))^\top, \tag{4}$$

$k = \operatorname{diag}(k_1, k_2, \ldots, k_N)$ and $x^0 := (x_1^0, \ldots, x_N^0)$.

In [6] the gain $k_i := kd_i$ for some $k > 0$ was chosen, i.e. (2) becomes the diffusive coupling

$$u_i = -k\sum_{j\in\mathcal{N}_i}(x_i - x_j), \quad i \in G, \tag{5}$$

or, equivalently,

$$u = -k\mathcal{L}x$$

and it was shown that practical synchronization can be achieved if k is sufficiently large; in fact for any $\varepsilon > 0$ there exists $K > 0$ such that for all $k \geq K$ it holds that

$$\limsup_{t\to\infty} |x_i(t) - s(t)| < \varepsilon,$$

where $s : [0,\infty) \to \mathbb{R}$ is the solution of the averaged dynamics

$$\dot{s} = \frac{1}{N}\sum_{i=1}^{N} f_i(t, s), \quad s(0) = \frac{1}{N}\sum_{i=1}^{N} x_i^0. \tag{6}$$

In particular, the agents synchronize practically and the resulting trajectory does neither depend on the graph topology nor on the specific chosen coupling strength k (as long as it is larger than a threshold), under the assumption that (6) is asymptotically stable. A problem in the above work is finding the threshold K because the algebraic connectivity of the network graph (the second smallest eigenvalue, say λ_2, of $\mathcal{L}$) and the growth of the nonlinear functions f_i affects K. The same difficulty arises for other consensus and synchronization problems, which are often handled by estimating λ_2 or employing adaptive technique for tuning the gain k; see, for example, [1], [8].

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In the context of adaptive control the above result is a typical "high gain" approach: choosing the feedback gain k sufficiently large leads to an arbitrarily accurate tracking of a desired trajectory, c.f. the survey [3]. Typical problems of high gain adaptive control are: 1) The necessary minimal value for the gain k is often not known and 2) a large gain k will in general amplify noise resulting in poor performance or even instability. Both problems are resolved with the so-called funnel controller, first introduced in [4]. The idea is rather simple: define a time-varying gain $k(\cdot)$ such that it is large when the error variable approaches the boundary of the allowed (time-varying) error region (the funnel) and small otherwise (i.e. when the error is already sufficiently small and no control action is required).

It is therefore a canonical idea to combine both approaches (high gain synchronization and funnel control) in order to synchronize heterogenous agents adaptively with the funnel controller. However, the combination is not straight-forward, because the desired trajectory for synchronization $s(\cdot)$ is not known to the individual agents (for this the agents need to know all dynamics of the other agents as well as the initial values), and only the *local* error is available for each agent (i.e. each agent compares its state value only with the state values of its neighbors). It should be noted that funnel control was used already for synchronization of connected agents in [2], but there synchronization was "enforced from the outside" by a given common reference trajectory, which all agents should follow and the interconnection just played the role of a bounded disturbance. In particular, the error for each agent was defined with respect to the externally given reference trajectory and not with respect to its neighbors.

In this note we propose two approaches for funnel synchronization (decentralized and weakly centralized) and illustrate the effectiveness by simulations. So far we could only prove a special case (weakly centralized for a sufficiently connected regular graph) and relaxing these (presumably too strong) assumptions is a topic of future research.

II. Funnel Synchronization

The major "ingredient" for funnel control is the desired time-varying error bound—the funnel $\mathcal{F}$—given by a scalar function $\varphi : [0,\infty) \to [\underline{\varphi}, \overline{\varphi}]$ with $\overline{\varphi} > \underline{\varphi} > 0$:

$$\mathcal{F} := \{\ (t,e)\ \mid |e| \le \varphi(t)\ \} \subseteq [0,\infty) \times \mathbb{R}$$

and the goal is to design a feedback law such that the error $t \mapsto e(t)$ evolves within $\mathcal{F}$ for all $t \ge 0$ (see Figure 1). In particular, the choice of the funnel boundary φ reflects the desired final accuracy of the synchronization as well as the speed of "convergence."

As mentioned in the introduction the idea of funnel control is to make the gain large whenever the error approaches the funnel boundary and a typical choice (other choices are possible, see [5]) is the reciprocal distance between the error and the funnel boundary:

$$k_\varphi(t,e) := \frac{1}{\varphi(t) - |e|}. \tag{7}$$

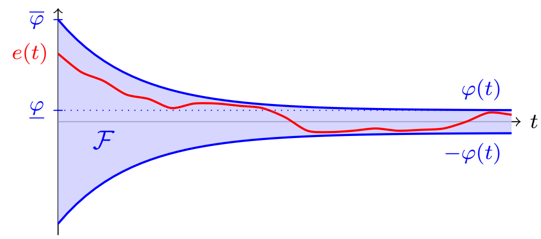


Fig. 1: The funnel $\mathcal{F}$.

We propose now two approaches for practical synchronization of heterogeneous agents via funnel control.

We call the feedback law

$$u_i(t) = -k_i(t)e_i(t), \tag{8}$$

where

$$k_i(t) := k_\varphi(t, e_i(t)),$$

decentralized funnel synchronization, i.e. each agent uses its own feedback gain based on the locally available error e_i given by (3). On the other hand we call the feedback law

$$u_i(t) = -k_{\max}(t) d_i e_i(t), \tag{9}$$

where

$$k_{\max}(t) := \max_{i \in G} k_\varphi(t, e_i(t))$$

weakly centralized funnel synchronization, i.e. each agent first calculates its own gain based on the local error and then all agents communicate their gain to the others and the maximal gain (scaled by the degree d_i of each agent) is applied. Note that in (9) the factor d_i is included in order to obtain a time-varying version of the diffusive coupling (5).

III. Simulations

Before presenting some preliminary theoretical results for funnel synchronization, we present and discuss some simulations, i.e. we "prove by example" that funnel synchronization works. For the simulation we consider agents as given in the simulations of [7], i.e. the i-th agent is governed by

$$\begin{aligned} f_i(t, x_i) &= (-1 + \delta_i)x_i + 10 \sin t \\ &\quad + 10 m_i^1 \sin(0.1t + \theta_i^1) + 10 m_i^2 \sin(10t + \theta_i^2), \end{aligned}$$

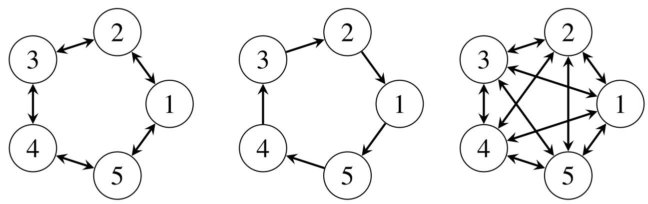
where δ_i, m_i^1, m_i^2 are independent random variables of standard normal distribution $N(0,1)$ and θ_i^1, θ_i^2 are independent random variables of uniform distribution on $[0, 2\pi]$. We consider $N = 5$ agents in a ring topology, i.e.

$$\mathcal{L} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}, \tag{10}$$

see also Figure 2a.

The initial value for each agent is chosen randomly with standard normal distribution $N(0,1)$.

The synchronization with diffusive coupling (5) and constant gain $k = 10$ is illustrated in Figure 3. Note that in all the following simulations the same (random) parameters are



(a) Undirected ring topology (b) Directed ring topology (c) Fully connected topology

Fig. 2: Different graph topology used in the simulation for $N = 5$ agents.

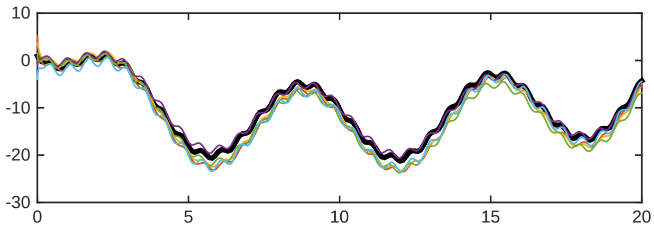


Fig. 3: Synchronization with constant gain $k = 10$ and $N = 5$ agents in an undirected ring topology. The thick black line is the solution of the averaged dynamics given by (6).

chosen (in this set of parameters, agent 2 had $\delta_2 > 1$, and so, it was even unstable before coupling).

As funnel boundaries we choose an exponentially decaying funnel boundary:

$$\varphi(t) = \underline{\varphi} + (\overline{\varphi} - \underline{\varphi})e^{-\lambda t}$$

with $\overline{\varphi} = 20$, $\underline{\varphi} = 1$, $\lambda = 1$. The simulation result for the decentralized funnel synchronization (8) is shown in Figure 4 for five agents connected via an undirected ring topology, i.e. with Laplacian (10).

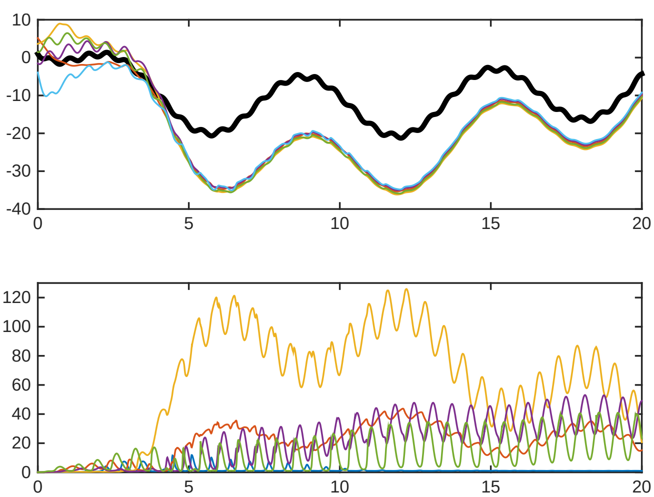


Fig. 4: **Decentralized funnel synchronization** (above) and the corresponding gains (below). The thick black line above is the solution of the averaged dynamics given by (6). Funnel parameters: $\overline{\varphi} = 20$, $\underline{\varphi} = 1$, $\lambda = 1$; $N = 5$ agents in an undirected ring topology.

Clearly, the agents synchronize practically, but not to the trajectory of the averaged dynamics. It turns out that the picture remains qualitively the same (with the same "limiting" trajectory) when a directed graph (see Figure 2b), instead of an undirected, is used (see Figure 5), when the funnel shape is changed (see Figure 6), or when the underlying graph is changed (see Figure 7) where the agents form a fully connected graph (see Figure 2c).

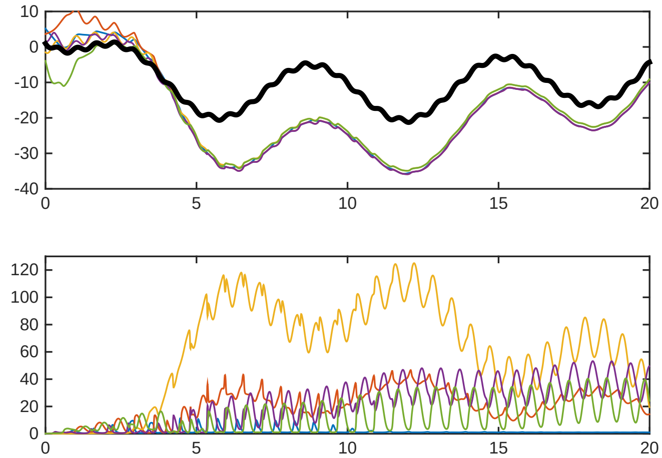


Fig. 5: Decentralized funnel synchronization (above) and the corresponding gains (below). The thick black line above is the solution of the averaged dynamics given by (6). Funnel parameters: $\overline{\varphi} = 20$, $\underline{\varphi} = 1$, $\lambda = 1$; $N = 5$ agents in a **directed** ring topology.

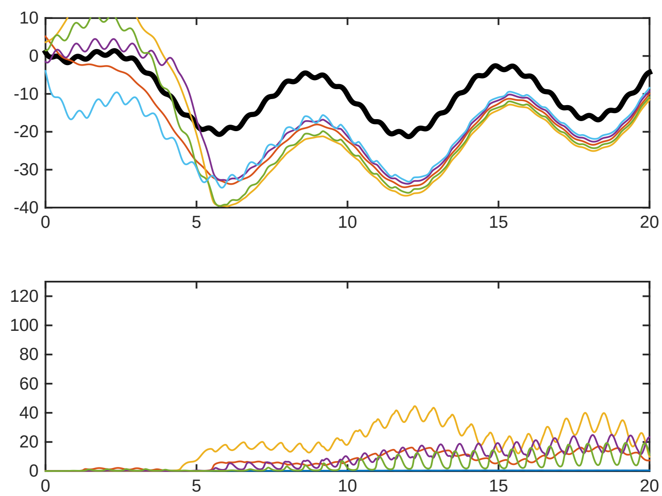


Fig. 6: Decentralized funnel synchronization (above) and the corresponding gains (below). The thick black line above is the solution of the averaged dynamics given by (6). **Funnel parameters**: $\overline{\varphi} = 30$, $\underline{\varphi} = 2$, $\lambda = 0.3$; $N = 5$ agents in ring topology.

What does however change the limiting trajectory is an additional amplification term in the gain formula:

$$k_\varphi(t, e) = \frac{\kappa}{\varphi(t) - |e|}.$$

Increasing κ results in convergence of the limit trajectory of the funnel controller towards the averaged trajectory, see Figure 8 for $\kappa = 200$.

Another possibility to approach the averaged dynamics given by (6) is to use the weakly centralized funnel synchronization (9). The simulation results are shown in Figure 9.

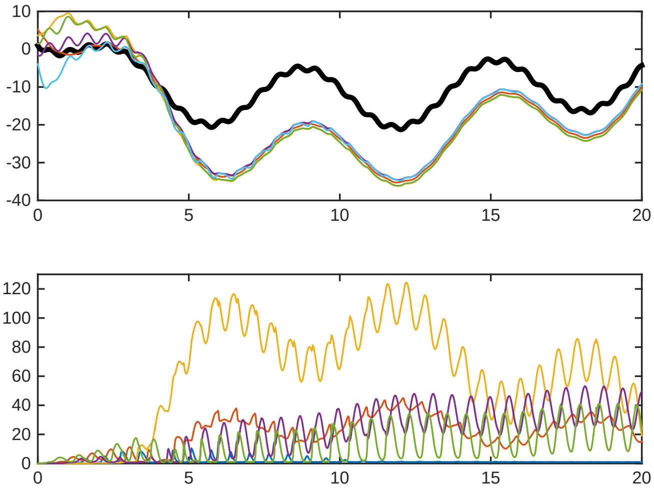


Fig. 7: Decentralized funnel synchronization (above) and the corresponding gains (below). The thick black line above is the solution of the averaged dynamics given by (6). Funnel parameters: $\overline{\varphi} = 20$, $\underline{\varphi} = 1$, $\lambda = 1$; $N = 5$ agents in **fully connected** topology.

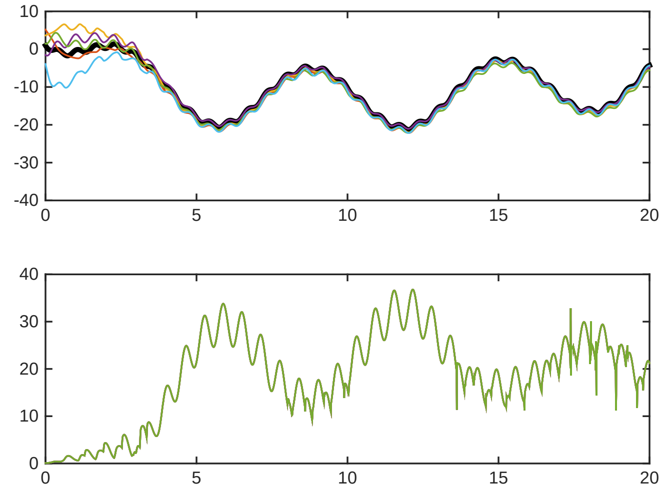


Fig. 9: **Weakly centralized funnel synchronization** (above) and the corresponding gain (below). The thick black line above is the solution of the averaged dynamics given by (6). Funnel parameters: $\overline{\varphi} = 20$, $\underline{\varphi} = 1$, $\lambda = 1$; $N = 5$ agents in an undirected ring topology.

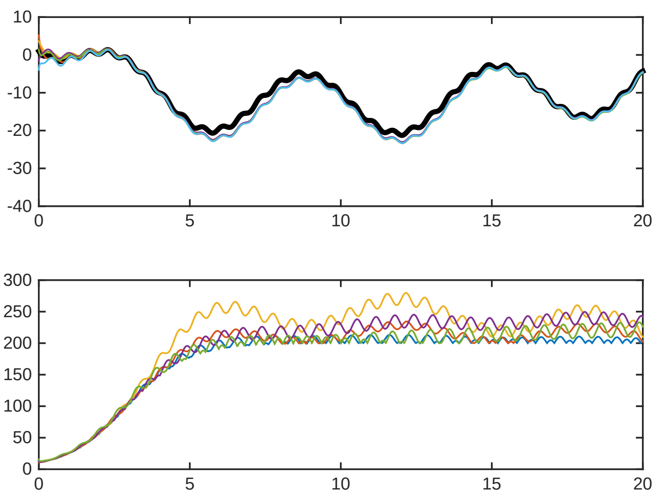


Fig. 8: Decentralized funnel synchronization (above) and the corresponding gains (below). The thick black line above is the solution of the averaged dynamics given by (6). Funnel parameters: $\overline{\varphi} = 20$, $\underline{\varphi} = 1$, $\lambda = 1$; $N = 5$ agents in ring topology; **additional funnel amplification** $\kappa = 200$.

IV. First preliminary theoretical results

A first step towards the proof that funnel synchronization works is the following lemma that shows that finite escape time cannot occur, provided the dynamics of the disconnected individual agents do not exhibit a certain finite escape time property:

Assumption 1: Let f be given by (4) and for

$$g : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R} : (t, \alpha) \mapsto \max_{\|z\|_2 = \sqrt{2\alpha}} z^\top f(t, z)$$

assume that

$$\dot{\alpha} = g(t, \alpha), \quad \alpha(0) \geq 0, \tag{11}$$

has a solution $\alpha : [0, \infty) \to \mathbb{R}_{\geq 0}$ for any nonnegative initial value (in particular, finite escape time does not occur).

Remark 1: Assumption 1 is stronger than the simple assumption that $\dot{x} = f(t, x)$ does not exhibit finite escape time. As a "counter example" consider $f_1(t, x_1) = (1 + \sin x_1)x_1^2$ and $f_2(t, x_2) = (1 - \sin x_2)x_2^2$. Clearly, $\dot{x} = f(t, x)$ does not exhibit finite escape time (because each of $x_1(t)$ and $x_2(t)$ will get stuck in some interval $(k\pi, (k+1)\pi)$ for some integer k). On the other hand, their averaged dynamics (6) is $\dot{s} = s^2$ whose solution with positive initial value exhibits finite escape time. Hence any feedback law which shows practical convergence towards the average dynamics (e.g. weakly centralized funnel control) also exhibits finite escape time, which of course is not desired. It is noted that Assumption 1 is not satisfied for this example.

Lemma 2 (No finite escape time): Consider a nonlinear system

$$\dot{x} = f(t, x) - M(t, x)x, \quad x(0) = x^0 \in \mathbb{R}^n \tag{12}$$

with f given by (4) satisfying Assumption 1 and some matrix function $M : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n \times n}$ for which $M(t, x)$ is positive semidefinite for any pair (t, x). Then any solution $x : [0, \omega) \to \mathbb{R}^n$ of (12) with $\omega > 0$ is bounded.

Proof: Let $x : [0, \omega) \to \mathbb{R}^n$ be a solution of (12), let $\alpha : [0, \infty) \to \mathbb{R}$ be a solution of (11) with initial condition $\alpha(0) = \frac{1}{2}\|x^0\|_2^2$ and let $C_\omega := \sup_{t \in [0,\omega)} \alpha(t) < \infty$ (because $\omega < \infty$ and α is continuous). Then

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}\|x(t)\|_2^2\right) &= x(t)^\top \dot{x}(t) \\
&= x(t)^\top f(t, x(t)) - \overbrace{x(t)^\top M(t, x(t)) x(t)}^{\geq 0} \\
&\leq \max_{\|z\|_2 = \|x(t)\|_2} z^\top f(t, z) = g\left(t, \tfrac{1}{2}\|x(t)\|_2^2\right).
\end{aligned}$$

Hence, for all $t \in [0, \omega)$

$$\tfrac{1}{2}\|x(t)\|_2^2 \leq \alpha(t),$$

in particular,

$$\|x(t)\|_2 \leq \sqrt{2C_\omega}.$$

∎

Assumption 2: The coupling graph (G, E) is undirected, connected and contains no self loops, i.e. $\mathcal{A}$ is symmetric, $\operatorname{rank} \mathcal{A} = N-1$ and $\mathcal{A}_{ii} = 0$ for all $i \in G$.

Corollary 3: Consider agents given by (1) whose dynamics satisfy Assumption 1 and whose network topology satisfies Assumption 2. Then under using weakly centralized funnel synchronization (9) the state variable cannot exhibit finite escape time.

Proof: Recall that the Laplacian of undirected graph is positive semidefinite. Then the claim follows from Lemma 2 as in the feedback loop the dynamics are given by

$$\dot{x} = f(t,x) - k_{\max}(t)\mathcal{D}\overline{\mathcal{L}}x$$

and $M(t, x(t)) := k_{\max}(t)\mathcal{D}\overline{\mathcal{L}} = k_{\max}(t)\mathcal{L}$ is positive semidefinite as a time-varying scalar multiple of the positive semidefinite Laplacian[1]. ■

In the following we will prove that the weakly centralized funnel synchronization works provided the following assumptions hold.

Assumption 3: The funnel boundary $\varphi : [0,\infty) \to [\underline{\varphi}, \overline{\varphi}]$ is continuously differentiable and non-increasing, and the initial values $x^0 \in \mathbb{R}^n$ of the agents satisfy:

$$\left| x_i^0 - \frac{1}{d_i}\sum_{j\in\mathcal{N}_i} x_j^0 \right| < \varphi(0), \quad \forall i = 1, 2, \ldots, N.$$

Assumption 4: The graph (G, E) is d-regular (i.e. $d_i = d$ for all $i \in G$) with

$$d > \frac{N}{2} - 1.$$

Assumption 5: Assume that $(t, x) \mapsto f(t, x)$ given by (4) is locally Lipschitz in x and measurable in t.

Theorem 4 (Weakly centralized funnel synchronization): Consider the multi-agent system (1) satisfying Assumptions 1-5. Then weakly centralized funnel synchronization (9) works, i.e. there exists a global solution $x : [0, \infty) \to \mathbb{R}^n$ and each agent's error evolves within the funnel, i.e. for all $i \in G$

$$|x_i(t) - \overline{x}_i(t)| \le \varphi(t), \quad \forall t \ge 0, \tag{13}$$

where $\overline{x}_i$ is given by (3). In particular, there exists $C > 0$ only depending on the graph topology such that for all $i, j \in G$

$$|x_i(t) - x_j(t)| \le C\varphi(t), \quad \forall t \ge 0, \tag{14}$$

i.e. practical synchronization is achieved because $\varphi(t)$ can be chosen as small as desired for large t.

Proof: Let

$$\Omega_\varphi := \left\{ \, (t,x) \in \mathbb{R}_{\ge 0} \times \mathbb{R}^n \;\middle|\; \|e\|_\infty = \|\overline{\mathcal{L}}x\|_\infty < \varphi(t) \, \right\}.$$

Since $f(t,x) - k_{\max}(t)\mathcal{D}\overline{\mathcal{L}}x$ is locally Lipschitz in x and measurable in t on Ω_φ, standard theory of ODEs (e.g. [11, Thm. 10.XX]) yields existence and uniqueness of a maximally extended solution $x : [0, \omega) \to \mathbb{R}^n$ such that $(t, x(t))$ belongs to Ω_φ. If $\omega = \infty$, then (13) follows. We will now show that $\omega < \infty$ leads to a contradiction.

Towards this end we show that the set

$$\Omega_{\varphi,\varepsilon} := \left\{ \, (t,x) \;\middle|\; t \in [0,\omega), \|\overline{\mathcal{L}}x\|_\infty \le \varphi(t) - \varepsilon \, \right\}$$

is positively invariant for sufficiently small positive ε such that

$$\varepsilon < \min\left\{ \frac{\underline{\varphi}}{2}, \frac{\underline{\varphi}}{2(C_{\tilde{f}} + C_{\dot{\varphi}})}, \frac{\varphi(0) - \max_{i\in G}|e_i(0)|}{2} \right\} \tag{15}$$

where $C_{\dot{\varphi}} := \sup_{t\in[0,\omega)} |\dot{\varphi}(t)|$ and

$$C_{\tilde{f}} := \max_{i\in G} \sup_{t\in[0,\omega)} \left| f_i(t, x_i(t)) - \frac{1}{d_i}\sum_{j\in\mathcal{N}_i} f_j(t, x_j(t)) \right| < \infty$$

which is well-defined since x is bounded in a finite time interval due to Corollary 3. Suppose that there exist $t_\varepsilon \in (0, \omega)$ and $i \in \{1, \ldots, N\}$ such that $\varphi(t_\varepsilon) - |e_i(t_\varepsilon)| = \varepsilon$ and $\varphi(t_\varepsilon) - |e_j(t_\varepsilon)| \ge \varepsilon$ for all $j \ne i$. Without restriction of generality we may further assume that $i = 1$. This implies that $k_{\max}(t_\varepsilon) = 1/\varepsilon$. Then

$$\begin{aligned}
\dot{e}_1 &= f_1(t, x_1) - k_{\max}(t) d_1 e_1 - \dot{\overline{x}}_1 \\
&= \tilde{f}_1(t,x) - k_{\max}(t) d_1 e_1 + \frac{1}{d_1}\sum_{j\in\mathcal{N}_1} k_{\max}(t) d_j e_j,
\end{aligned}$$

where $\tilde{f}_1(t,x) := f_1(t,x_1) - \overline{f}_1(t,x) := f_1(t,x_1) - \frac{1}{d_1}\sum_{j\in\mathcal{N}_1} f_j(t,x_j)$. Since (G,E) is assumed to be d-regular, the expression for $\dot{e}_1$ simplifies to

$$\dot{e}_1 = \tilde{f}_1(t,x) - k_{\max}(t)\left(d e_1 - \sum_{j\in\mathcal{N}_1} e_j \right).$$

For any undirected graph (G, E) we have that

$$(d_1, d_2, \ldots, d_N)\overline{\mathcal{L}} = 1_N^\top \mathcal{L} = 0,$$

in particular,

$$(d_1, d_2, \ldots, d_N)e = (d_1, d_2, \ldots, d_N)\overline{\mathcal{L}}x = 0.$$

For the d-regular case this implies that

$$\sum_{j\in\mathcal{N}_1} e_j = -e_1 - \sum_{j\notin\mathcal{N}_1\cup\{1\}} e_j$$

and we arrive at

$$\dot{e}_1 = \tilde{f}_1(t,x) - k_{\max}(t)\left((d+1)e_1 + \sum_{j\notin\mathcal{N}_1\cup\{1\}} e_j \right).$$

Assume that $e_1(t_\varepsilon) \ge 0$ (the case $e_1(t_\varepsilon) \le 0$ can be treated completely analogously). Then, invoking $e_1(t_\varepsilon) \ge |e_j(t_\varepsilon)|$ for all j,

$$\dot{e}_1(t_\varepsilon) \le C_{\tilde{f}} - \frac{1}{\varepsilon}(d+1)e_1(t_\varepsilon) + \frac{1}{\varepsilon}(N-d-1)e_1(t_\varepsilon)$$

[1]Note that in general $K\mathcal{L}$ is *not* necessarily positive semidefinite for some positive semidefinite matrix $\mathcal{L}$ and some positive diagonal matrix K, hence Lemma 2 cannot be used in the decentralized case where K is not a positive multiple of the identity matrix.

because the cardinality of $\mathcal{N}_1 \cup \{1\}$ is $d+1$. This in turn implies that, by Assumption 4 with the fact that N and d are integers,

$$\begin{aligned}\dot{e}_1 &\leq C_{\tilde{f}} - \frac{1}{\varepsilon}(2d-(N-2))e_1(t_\varepsilon) \\ &\leq C_{\tilde{f}} - \frac{1}{\varepsilon}e_1(t_\varepsilon).\end{aligned}$$

Since $\underline{\varphi}/2 > \varepsilon = \varphi(t_\varepsilon) - e_1(t_\varepsilon) \geq \underline{\varphi} - e_1(t_\varepsilon)$ we can conclude, by (15), that

$$\dot{e}_1(t_\varepsilon) \leq C_{\tilde{f}} - \frac{\underline{\varphi}}{2\varepsilon} < -C_{\dot{\varphi}} \leq \dot{\varphi}(t_\varepsilon).$$

So, the distance $\varphi(t) - e(t)$ between the funnel boundary φ and the error e_1 has a positive derivative at $t = t_\varepsilon$, i.e. it increases at t_ε, which shows that the set $\Omega_{\varphi,\varepsilon}$ is positively invariant. In particular, the solution x has a positive distance from the boundary of Ω_φ, and thus, $\omega = \infty$ and (13) follows.

In order to obtain (14) from (13), let R be a $N \times (N-1)$ matrix such that $[1_N, R]$ becomes nonsingular, and, for each x, there are $\xi_0 \in \mathbb{R}$ and $\bar{\xi} \in \mathbb{R}^{N-1}$ such that $x = 1_N\xi_0 + R\bar{\xi}$. Since the graph is connected, 0 is the simple eigenvalue of $\overline{\mathcal{L}}$ and nullity of $\overline{\mathcal{L}}$ is 1 with a basis of $\{1_N\}$, so that $\overline{\mathcal{L}}R =: A$ has full column rank. Let $\delta_{i,j}$ be a row vector whose i-th element is 1, j-th element is -1, and other elements are 0. Then, since $e = \overline{\mathcal{L}}x = \overline{\mathcal{L}}(1_N\xi_0 + R\bar{\xi}) = A\bar{\xi}$ and $\|e\| \leq \sqrt{N}\varphi$ by (13),

$$\begin{aligned}|x_i - x_j| &= |\delta_{i,j}x| = |\delta_{i,j}1_N\xi_0 + \delta_{i,j}R\bar{\xi}| = |\delta_{i,j}R\bar{\xi}| \\ &\leq \max_{i,j\in G} \|\delta_{i,j}R\|\|\bar{\xi}\| = \max_{i,j\in G} \|\delta_{i,j}R\|\|(A^TA)^{-1}A^Te\| \\ &\leq \max_{i,j\in G} \|\delta_{i,j}R\|\|(A^TA)^{-1}A^T\|\sqrt{N}\varphi =: C\varphi\end{aligned}$$

which shows (14). ∎

Since the ring topology with $N = 5$ agents is d-regular with $d = 2 > \frac{N}{2} - 1 = 1.5$, our Theorem 4 shows that weakly centralized funnel synchronization works for the example also studied in [7], i.e. Figure 9 is not a coincidence. In fact, our result generalizes the stability result of [7] significantly, because we do neither need to assume that the averaged dynamics (6) are bounded nor that $\frac{\partial f_i(t,x_i)}{\partial x_i}$ is bounded; however, in that case we cannot guarantee boundedness of the gain $k(\cdot)$.

We expect that also the other simulation results shown in Figures 4–8 are not a coincidence, i.e. in particular we should be able to relax Assumption 4 and also allow for a decentralized control. A possible approach to handle the more general case might be contraction analysis [9], which was already successfully applied in the context of synchronization [10].

V. Conclusion

A preliminary result is obtained that utilizes funnel control to the problem of practical consensus for heterogeneous agents. This approach (which we call funnel synchronization) can provide some required amount of high gain automatically, one need not know the required threshold of the gain for consensus. Moreover, the high gain effect applies only when it is necessary; that is, if all the trajectories are already close to each other then the coupling gain becomes small, which is another benefit of funnel synchronization.

Our repeated simulation study shows that the aforementioned benefits of funnel synchronization apply for most cases of decentralized synchronization schemes under general connected network graphs. In this paper, however, we have presented rigorous arguments just for weakly centralized funnel synchronization under a regular graph.

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