

Switch observability for a class of inhomogeneous switched DAEs

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Abstract—Necessary and sufficient conditions for switching time and switch observability of a class of inhomogeneous switched differential algebraic equations (DAEs) are obtained. A characterization of initial states and inputs for which switched DAEs are switch unobservable is also provided by using the zeros of an augmented system obtained by combining the output of two modes suitably.

I. INTRODUCTION

Switch observability in switched system refers to identifying the switching signal from the knowledge of the input and the output. It can be used in fault detection [1], identifying link connections and disconnections in networked systems [2], designing observers for switched systems [3], [4], [5], [6], [7]. In [7] an observer design for switched differential algebraic equations (DAEs) was considered which required the knowledge of which mode the system is currently running in. As a result, detecting the switching signal from the input and output becomes essential for realizing this observer. In this paper, we characterize the initial conditions and inputs for which it is possible to reconstruct the switching signal. This information can then be used for realizing the observer in [7].

We consider two problems, namely, mode detection and identification of the switching instants. Mode detection is feasible, if it is possible to distinguish between outputs of different subsystems, when subjected to the same input. In this paper, we provide a complete characterization of initial conditions and analytic inputs under which it is possible to distinguish between different subsystems. Detecting switching instants require that at the switching instant, the output will behave in a manner so that the switch is apparent. Loss of analyticity of the output at the switching instant for switched systems subject to analytic inputs indicates the change of mode. We also provide a complete characterization of initial conditions and inputs for which switching times can be detected. We will use strong observability notions defined in [8] for switched ordinary differential equations (ODEs) namely, strong σ -, t_S - and σ_1 -observability. These notions are strong in the sense that the observability holds for all inputs as opposed to standard notion which require observability for “almost all” inputs (generically) [9], [10], [11]. In [12] strong σ - and t_S -observability for discrete time ODE-system is characterized.

In [8] the switch observability problem was studied for switched homogeneous and inhomogeneous ODEs and then in [13] for switched homogeneous DAEs. Here we consider switch observability for a class of inhomogeneous switched DAEs. Our contribution is twofold, 1) we give necessary and sufficient conditions for switch observability of a class of switched inhomogeneous DAEs (with modes that are strictly proper systems¹) which are generalizations of conditions provided in [13], [8] and 2) we give a characterization of initial conditions and inputs for which we loose switch observability. Switched DAEs provide richer dynamics compared to ODEs, in terms of appearance of jumps, Dirac impulse and its derivatives in the output². The conditions derived in this paper use this additional information to give necessary and sufficient conditions for mode detection and switching time observability.

This paper is arranged as follows. In Section II we formulate the problem by defining observability notions. Further, in Section III, we introduce the notation that will be used and assumptions under which we will be operating. We also discuss in brief the distributional solution concept for switched DAEs which will be used throughout the paper. Our main results are presented in Section IV, where we provide necessary and sufficient conditions for strong σ -, t_S - and σ_1 -observability. Due to space constraints we omit the proofs.

II. PROBLEM FORMULATION

We consider switched DAEs of the form

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u, \\ y &= C_\sigma x, \end{aligned} \quad (1)$$

where switching is governed by a switching function $\sigma : \mathbb{R} \rightarrow \mathcal{P} := \{1, \dots, p\}$, $p \in \mathbb{N}$, which is piecewise constant and right continuous. On each time interval where σ is constantly p the dynamics are governed by a DAE

$$\Sigma_p : \begin{aligned} E_p \dot{x} &= A_p x + B_p u, \\ y &= C_p x, \end{aligned}$$

with $E_p, A_p \in \mathbb{R}^{n \times n}$, $B_p \in \mathbb{R}^{n \times n_u}$, $C_p \in \mathbb{R}^{n_y \times n}$. We assume that every matrix pair (E_p, A_p) , $p \in \mathcal{P}$, is regular, i.e. $\det(sE_p - A_p)$ is not the zero polynomial. For a given switching signal, the set of switching times is given by $T_\sigma := \{t \in \mathbb{R} \mid \sigma(t^-) \neq \sigma(t^+)\}$. We will assume that all switches occur after the initial time $t = 0$, i.e. $T_\sigma \subseteq \mathbb{R}_{>0}$.

¹See Assumption (A3) in Section IV-B. Relaxing this assumption is currently under investigation.

²This is true even with Assumption (A3) in place.

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(which is usually the case), then the last rows of $\bar{y}[t_p]$ are all zero and do not contain any information. Nevertheless, we use the definition (6) instead of $\bar{y}[t_p] = -\mathbf{O}^{\text{imp},[\nu_p]}x(t_p^-)$ in order to have a mode-independent dimension of $\bar{y}[t_p]$.

IV. OBSERVABILITY CHARACTERIZATIONS

A. Observability Definitions

For $u = 0$, $x_0 = 0$ we have $y_{(x_0, \sigma, u)} = 0$ for any switching signal σ . Hence we cannot expect to determine the switching signal in this case. Similarly, if $u = 0$ and the state jumps to zero at a switch, i.e. $x(t_S^+) = 0$, we cannot expect to determine the switching signal from there onwards. To get a useful notion of switching signal observability, we thus have to exclude such cases. In the recent works [13], [8] different approaches were used: For homogeneous ODEs, it was sufficient to exclude zero initial states. In the inhomogeneous case, we could either make certain assumptions to ensure that the solution stays nonzero or consider equivalence classes of switching signals. For homogeneous switched DAEs, it was also necessary to consider equivalence classes of switching signals.

Here we follow another, more intuitive approach: We define an interval \mathcal{I} on which we want to determine the switching signal. For an initial value x_0 , switching signal σ and smooth input u we define the *essential interval* $\mathcal{I}_{(x_0, \sigma, u)}$ as

$$\mathcal{I}_{(x_0, \sigma, u)} = \left\{ t \mid \begin{array}{l} \exists i \in \mathbb{N} : u^{(i)}(t) \neq 0 \vee \\ x_{(x_0, \sigma, u)}(t^+) \neq 0 \vee x_{(x_0, \sigma, u)}[t] \neq 0 \end{array} \right\}.$$

In general, $\mathcal{I}_{(x_0, \sigma, u)}$ may be a union of intervals, to avoid this we make the following assumption:

(A1) The input is real analytic.

We then have the following result.

Lemma 2: Consider a regular switched DAE (1) satisfying (A1), then for all (consistent) initial values x_0 and all switching signals σ we have

$$\begin{aligned} \mathcal{I}_{(x_0, \sigma, u)} &= \mathbb{R}, & \text{for } u \neq 0, \\ \mathcal{I}_{(x_0, \sigma, u)} &= (-\infty, T) \text{ or } (-\infty, T], & \text{for } u = 0, \end{aligned}$$

for some $T \in T_\sigma \cup \{-\infty, \infty\}$.

Similar as in [13] we make the following assumption on the input matrices in (1):

(A2) For any modes $p, q \in \mathcal{P}$, $p \neq q$: $\ker \begin{bmatrix} B_p \\ B_q \end{bmatrix} = \{0\}$.

To see the need for (A2), assume that it does not hold, i.e. that there exists modes $p \neq q$ and $u \neq 0$ analytic with $B_p u = B_q u = 0$. Then $y_{(0, p, u)} = y_{(0, q, u)} = 0$ and $\mathcal{I}_{(0, p, u)} = \mathbb{R}$ imply that the mode of the system cannot be observable. Assumption (A2) is needed to deal with intervals of zero state. It is possible to avoid (A2) by introducing an equivalence on switching signals, see [13].

For $u = 0$, the solution might go to zero and stay zero, making further switches undetectable. In [8] we have seen that this leads to an extra condition (cf. the forthcoming condition (11) in Theorem 5) and an additional term $\mathcal{M}_{i,j,p,q}$ in the characterization of strong σ_1 -observability (cf. the

forthcoming Theorem 9). For $u \neq 0$, however, such problems do not occur which simplifies the subsequent analysis.

We will now formally define the observability notions studied in this paper:

Definition 3: The switched DAE (1) is called

- strongly σ -observable if, and only if, for all $(\sigma, \tilde{\sigma})$, all $x_0 \in \mathcal{V}_{\sigma(0)}^*$, $\tilde{x}_0 \in \mathcal{V}_{\tilde{\sigma}(0)}^*$

$$\sigma \neq \tilde{\sigma} \text{ on } \mathcal{I}_{(x_0, \sigma, u)} \implies y_{(x_0, \sigma, u)} \neq y_{(\tilde{x}_0, \tilde{\sigma}, u)} \quad (7)$$

holds for all analytic u , i.e. the switching signal can be determined (on the essential interval) from the knowledge of the output and the input (for all possible inputs and all initial states);

- strongly (x, σ) -observable if, and only if, for all $(\sigma, \tilde{\sigma})$, all $x_0 \in \mathcal{V}_{\sigma(0)}^*$, $\tilde{x}_0 \in \mathcal{V}_{\tilde{\sigma}(0)}^*$

$$\left(\begin{array}{l} \sigma \neq \tilde{\sigma} \text{ on } \mathcal{I}_{(x_0, \sigma, u)} \\ \vee x_0 \neq \tilde{x}_0 \end{array} \right) \implies y_{(x_0, \sigma, u)} \neq y_{(\tilde{x}_0, \tilde{\sigma}, u)} \quad (8)$$

holds for all analytic u , i.e. the switching signal and the state can be determined (on the essential interval) from the knowledge of the output and the input (for all possible inputs and initial states);

- strongly σ_1 -observable if and only if for all $x_0 \in \mathcal{V}_{\sigma(0)}^*$, $\tilde{x}_0 \in \mathcal{V}_{\tilde{\sigma}(0)}^*$, all analytic u and all $(\sigma, \tilde{\sigma})$ with σ having at least one switch in $\mathcal{I}_{(x_0, \sigma, u)}$ the implication (7) holds. This means the switching signal can be determined from the knowledge of the input and the output provided at least one essential switch has occurred;

- strongly (x, σ_1) -observable if and only if for all $x_0 \in \mathcal{V}_{\sigma(0)}^*$, $\tilde{x}_0 \in \mathcal{V}_{\tilde{\sigma}(0)}^*$, all analytic u and all $(\sigma, \tilde{\sigma})$ with σ having at least one switch in $\mathcal{I}_{(x_0, \sigma, u)}$ the implication (8) holds. This means the switching signal and the state can be determined from the knowledge of the input and the output provided at least one essential switch has occurred;

- strongly t_S -observable if and only if for all $(\sigma, \tilde{\sigma})$, all $x_0 \in \mathcal{V}_{\sigma(0)}^*$, $\tilde{x}_0 \in \mathcal{V}_{\tilde{\sigma}(0)}^*$ and all analytic u it holds

$$T_\sigma \neq T_{\tilde{\sigma}} \text{ on } \overline{\mathcal{I}_{(x_0, \sigma, u)}} \implies y_{(x_0, \sigma, u)} \neq y_{(\tilde{x}_0, \tilde{\sigma}, u)},$$

i.e. the essential switching times can be determined from the knowledge of the input and the output. We want to detect the switching times in the closure of the essential interval, because at the boundary of the essential interval the non-zero state jumps to zero and this jump should be detectable in the output (independently of the possible presence of Dirac impulses).

Note that we consider *strong* observability notions and not a generic (or weak) notion where observability is only required for almost all (or one) input signal.

With the same arguments as in [13, Lem. 2] the (surprising) equivalence between strong σ - and strong (x, σ) -observability and between strong σ_1 - and strong (x, σ_1) -observability can be shown, hence we will focus in the following on σ - and σ_1 -observability. Furthermore, it is clear that strong σ -observability is sufficient for strong σ_1 -observability which in turn is sufficient for t_S -observability, however the converse is not true in general.

B. σ -observability

For the switched DAE (1) and two modes $i, j \in \mathcal{P}$ consider first the following augmented system

$$\Sigma_{(i,j)} : \begin{bmatrix} E_i & 0 \\ 0 & E_j \end{bmatrix} \dot{\xi} = \begin{bmatrix} A_i & 0 \\ 0 & A_j \end{bmatrix} \xi + \begin{bmatrix} B_i \\ B_j \end{bmatrix} u, \quad (9)$$

where $\xi = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$. The role of the augmented systems for σ -observability is expressed by the following Lemma.

Lemma 4: For the switched system (1) the two constant switching signals $\sigma \equiv i$ and $\tilde{\sigma} \equiv j$ can be distinguished for any initial values and common input on the essential interval if, and only if, the augmented system $\Sigma_{(i,j)}$ is unknown-input-(ui)-observable (see Appendix).

In order to characterize ui-observability of the augmented system $\Sigma_{(i,j)}$ we make the following assumption:

(A3) The transfer matrix from u to x for each DAE Σ_p given by $(sE_p - A_p)^{-1}B_p$ is strictly proper, or, equivalently, $B_p^{\text{imp}} = 0$ for all $p \in \mathcal{P}$.

Assumption (A3) means that in the QWF (2) the input is not affecting the nilpotent part. In other words there is no feedthrough from u (or its derivatives) to x . Lemma 11 in the Appendix together with (A3) now shows that

$$\text{rk} \begin{bmatrix} \mathcal{O}_i^{[2n]} & \mathcal{O}_j^{[2n]} & \Gamma_i^{[2n]} - \Gamma_j^{[2n]} \\ \mathcal{V}_i^* & \mathcal{V}_j^* & \Gamma_i^{[2n]} - \Gamma_j^{[2n]} \end{bmatrix} = \dim \mathcal{V}_i^* + \dim \mathcal{V}_j^* + \text{rk} \left(\Gamma_i^{[2n]} - \Gamma_j^{[2n]} \right) \quad (10)$$

characterizes ui-observability. Hence we already have derived a *necessary* condition for σ -observability. In contrast to switched ODEs condition (10) is however *not sufficient*. This can already be seen in the homogeneous case, see e.g. [8, Ex. 7]. In fact, only for $u = 0$ the condition (10) is not sufficient for σ -observability, because $u \neq 0$ already implies that the essential interval is the whole real axis and ui-observability of the augmented systems $\Sigma_{(i,j)}$ for all $i, j \in \mathcal{P}$ with $i \neq j$ implies the ability to determine the active mode on all open intervals between the switching times, which implies that σ can be determined. Hence by recalling the condition

$$\mathcal{V}_i^* \cap \mathcal{W}_j^* \cap \mathcal{W}_k^* \cap \ker \left(\mathcal{O}_i^{\text{imp}} - \mathcal{O}_j^{\text{imp}} \right) \subseteq \ker \begin{bmatrix} E_j^{\text{imp}} \\ E_k^{\text{imp}} \end{bmatrix} \quad (11)$$

derived in [8] for homogeneous switched DAEs we have our first main result:

Theorem 5: The switched DAE (1) satisfying (A1), (A2) and (A3) is strongly σ -observable, or equivalently, strongly (x, σ) -observable if, and only if, (10) and (11) hold for all pairwise different $i, j, k \in \mathcal{P}$.

In case the switched DAE is not strongly σ -observable, we now want to investigate the (nontrivial) inputs and initial conditions which lead to σ -unobservability.

Lemma 6: Consider $\lambda \in \mathbb{C}$ for which there exists $\hat{x}_0 \in \mathbb{C}^n, \hat{\tilde{x}}_0 \in \mathbb{C}^n, u_0 \in \mathbb{C}^m$ such that

$$\begin{bmatrix} \lambda E_i - A_i & 0 & -B_i \\ 0 & \lambda E_j - A_j & -B_j \\ C_i & -C_j & 0 \end{bmatrix} \begin{pmatrix} \hat{x}_0 \\ \hat{\tilde{x}}_0 \\ u_0 \end{pmatrix} = 0. \quad (12)$$

Let $u(t) = e^{\text{Re } \lambda t} (\cos(\text{Im } \lambda t) \text{Re } u_0 - \sin(\text{Im } \lambda t) \text{Im } u_0)$. Then $y_{(x_0, p, u)}(t) = y_{(\hat{x}_0, q, u)}(t)$ for all $t \geq 0$ for initial conditions $x_0 = \text{Re } \hat{x}_0$ and $\tilde{x}_0 = \text{Re } \hat{\tilde{x}}_0$.

Lemma 6 gives us a way to compute the initial conditions and inputs which lead to σ -unobservability. Consider a matrix pencil

$$P_{(i,j)}(\lambda) = \begin{bmatrix} \lambda E_i - A_i & 0 & -B_i \\ 0 & \lambda E_j - A_j & -B_j \\ C_i & -C_j & 0 \end{bmatrix}$$

Then its generalized eigenstructure gives us inputs and state-trajectory for which we lose σ -observability for $\sigma = i$ and $\tilde{\sigma} = j$. The generalized eigenstructure of $P_{(i,j)}(\lambda)$ can be computed by using quasi-Kronecker form (see [15] and [14]).

Example 1: Consider the switched DAE (1) with $\mathcal{P} = \{1, 2, 3\}$ and

$$\begin{aligned} E_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, & A_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, & B_1 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, & C_1 &= [-1 \ 0 \ 0] \\ E_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & A_2 &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, & B_2 &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, & C_2 &= [-\frac{1}{2} \ -\frac{1}{2} \ 0] \\ E_3 &= E_2, & A_3 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, & B_3 &= B_2, & C_3 &= [1 \ -2 \ 0] \end{aligned}$$

This system is not σ -observable because (10) is not satisfied for mode pairs (1, 3) and (2, 3). In Table I, we list all the initial conditions and inputs for which the considered switched DAE is not σ -observable. These initial conditions and inputs are computed by using matrix pencils $P_{(1,2)}(\lambda)$, $P_{(2,3)}(\lambda)$, and $P_{(1,3)}(\lambda)$ created for the pairs (1, 2), (2, 3) and (1, 3) respectively.

C. t_S -observability

Note that, by definition, t_S -observability is equivalent to the ability to determine the switching time t_S from the output. Assume that at time t_S , the mode changes from i to j for some $i, j \in \mathcal{P}$ i.e.,

$$\sigma = \begin{cases} i, & t < t_S, \\ j, & t \geq t_S. \end{cases}$$

Then, the switching time t_S can be detected from the output in two ways namely, 1) detecting an impulse in output at t_S

TABLE I

EXAMPLE 1: LIST OF INITIAL CONDITIONS AND INPUTS THAT LEAD TO σ -UNOBSERVABILITY

Pencil	$P_{(1,2)}(\lambda)$	$P_{(2,3)}(\lambda)$	$P_{(1,3)}(\lambda)$
Singular for	no $\lambda \in \mathbb{C}$	$\lambda = \frac{1}{2} \pm i \frac{\sqrt{3}}{2}$	$\lambda = 0$
Initial Conditions	none	$x_0 = (-u_0, 0)$, $\tilde{x}_0 = (-\frac{1}{2}u_0, -\frac{1}{2}u_0, 0)$	$x_0 = (c, 0, 0)$ $\tilde{x}_0 = (-c, 0, 0)$
Input	none	$u(t) = e^{\frac{t}{2}} \left(-\frac{1}{2} \cos\left(\frac{\sqrt{3}t}{2}\right) - \frac{\sqrt{3}}{6} \sin\left(\frac{\sqrt{3}t}{2}\right) \right) u_0$	$u(t) \equiv 0$

i.e., $y[t_S] \neq 0$ or 2) the output is not smooth at t_S i.e., there exists $l \in \mathbb{N}$ such that $y^{[l]}(t_S^-) \neq y^{[l]}(t_S^+)$.

For characterizing strong t_S -observability, we need a notion of the set of controllable weakly unobservable states of system Σ (see (16)) denoted as $\mathcal{R}(\Sigma)$ (see [19]).

$$\mathcal{R}(\Sigma) := \left\{ x_0 \in \mathcal{V}^* \mid \begin{array}{l} \exists u(\cdot), T > 0 : y(x_0, u) \equiv 0 \\ \text{and } x(x_0, u)(T) = 0 \end{array} \right\}.$$

This notion enables us to characterize strong t_S -observability as follows:

Lemma 7: A switched DAE (1) satisfying (A1), (A2) and (A3) is strongly t_S -observable if and only if it holds

$$\mathcal{R}(\Sigma_{i,j}) = \{0\} \quad (13)$$

and

$$\begin{aligned} \text{rank} \begin{bmatrix} \mathcal{O}_i^{[2n]} - \mathcal{O}_j^{[2n]} \Pi_i & \Gamma_i^{[2n]} - \Gamma_j^{[2n]} \\ \mathbf{O}_j^{\text{imp}} \Pi_i & 0 \end{bmatrix} & \quad (14) \\ = \dim \mathcal{V}_i^* + \text{rank} \left(\Gamma_i^{[2n]} - \Gamma_j^{[2n]} \right), & \end{aligned}$$

hold for all $i \neq j$.

We relate the conditions to those obtained in [13] and [8]. (13) generalizes a corresponding condition for inhomogeneous switched ODEs. As apparent from the definition of $\mathcal{R}(\Sigma)$, it is not required in the homogeneous case. (14) generalizes both the conditions for inhomogeneous switched ODEs and homogeneous switched DAEs. The terms $\mathbf{O}_j^{\text{imp}}$ and Π_i in (14) correspond to an impulse and state jump, respectively. Hence they do not appear for ODEs. In the homogeneous DAE case, the second column block (as well as the term $\text{rank}(\Gamma_i^{[2n]} - \Gamma_j^{[2n]})$) will not appear.

Remark 8: By defining the set of weakly unobservable states as

$$\mathcal{W}(\Sigma) := \{ x_0 \in \mathcal{V}^* \mid \exists u(\cdot) : y(x_0, u) \equiv 0 \}.$$

we see that condition (14) is in fact equivalent to

$$\mathcal{W}(\Sigma_{(i,j)}) \cap \begin{bmatrix} I \\ \Pi_i \end{bmatrix} \ker \mathbf{O}_j^{\text{imp}} \Pi_i = \{0\}.$$

In particular, the initial value pairs $\begin{pmatrix} x_0 \\ \bar{x}_0 \end{pmatrix}$ leading to t_S -unobservability are exactly those which are in the set $\left(\mathcal{W}(\Sigma_{(i,j)}) \cap \text{im} \begin{bmatrix} I \\ \Pi_i \end{bmatrix} \ker \mathbf{O}_j^{\text{imp}} \Pi_i \right) \cup \mathcal{R}(\Sigma_{i,j})$.

D. σ_1 -observability

The notion of σ_1 -observability is a weaker version of σ -observability. For σ -observability, the information obtained at the switching instants is not utilized, since we are comparing the output of systems for two distinct but constant switching signals. Instead of comparing two constant switching signals, two distinct switching signals with one of them having at least one switch are compared for σ_1 -observability. Note that by definition strong t_S -observability is necessary for strong σ_1 -observability.

To obtain a characterization for σ_1 -observability, we compare two switching signals

$$\sigma(t) = \begin{cases} i, & t < T_{ij}, \\ j, & t \geq T_{ij}. \end{cases} \quad \text{and} \quad \tilde{\sigma}(t) = \begin{cases} p, & t < T_{pq}, \\ q, & t \geq T_{pq}. \end{cases}$$

Due to strong t_S -observability, it turns out that we only need to consider the case $T_{ij} = T_{pq} = t_S$. The subsequent necessary and sufficient condition for σ_1 -observability has to be satisfied for all $i, j, p, q \in \mathcal{P}$ given that $\sigma \neq \tilde{\sigma}$. As a result, we end up comparing augmented systems $\Sigma_{(i,p)}$ and $\Sigma_{(j,q)}$ for strong σ -observability. Thus, to examine the σ_1 -observability, we consider ui-observability of the bigger augmented system $\Sigma_{i,j,p,q}$ formed by combining $\Sigma_{(i,p)}$ and $\Sigma_{(j,q)}$ which is defined as follows

$$\begin{aligned} \begin{bmatrix} E_i & 0 & 0 & 0 \\ 0 & E_j & 0 & 0 \\ 0 & 0 & E_p & 0 \\ 0 & 0 & 0 & E_q \end{bmatrix} \dot{\xi} &= \begin{bmatrix} A_i & 0 & 0 & 0 \\ 0 & A_j & 0 & 0 \\ 0 & 0 & A_p & 0 \\ 0 & 0 & 0 & A_q \end{bmatrix} \xi + \begin{bmatrix} B_i \\ B_j \\ B_p \\ B_q \end{bmatrix} u, \\ \chi &= \begin{bmatrix} C_i & 0 & -C_p & 0 \\ 0 & C_j & 0 & -C_q \end{bmatrix} \xi + \begin{bmatrix} D_i - D_p \\ D_j - D_q \end{bmatrix} u. \end{aligned}$$

For ui-observability of $\Sigma_{i,j,p,q}$ it suffices to consider the matrices $\mathcal{O}_{i,j,p,q}^{[4n]}$ and $\Gamma_{i,j,p,q}^{[4n]}$. Note that these matrices are given by

$$\begin{aligned} \mathcal{O}_{i,j,p,q}^{[\nu]} &= T \begin{bmatrix} \mathcal{O}_i^{[\nu]} & -\mathcal{O}_p^{[\nu]} & 0 & 0 \\ 0 & 0 & \mathcal{O}_j^{[\nu]} & -\mathcal{O}_q^{[\nu]} \end{bmatrix}, \\ \Gamma_{i,j,p,q}^{[\nu]} &= T \begin{bmatrix} \Gamma_i^{[\nu]} - \Gamma_j^{[\nu]} \\ \Gamma_p^{[\nu]} - \Gamma_q^{[\nu]} \end{bmatrix} \end{aligned}$$

for some permutation matrix T .

Theorem 9: A switched DAE (1) satisfying (A1), (A2) and (A3) is strongly σ_1 -observable if and only if it is strongly t_S -observable and satisfies

$$\begin{aligned} \text{rank} \begin{bmatrix} \mathcal{O}_i^{[4n]} & \mathcal{O}_p^{[4n]} & \Gamma_i^{[4n]} - \Gamma_p^{[4n]} \\ \mathcal{O}_j^{[4n]} \Pi_i & \mathcal{O}_q^{[4n]} \Pi_p & \Gamma_j^{[4n]} - \Gamma_q^{[4n]} \\ \mathbf{O}_j^{\text{imp}} \Pi_i & \mathbf{O}_q^{\text{imp}} \Pi_p & 0 \end{bmatrix} & \quad (15) \\ = \dim \mathcal{V}_i^* + \dim \mathcal{V}_p^* + \text{rank} \begin{bmatrix} \Gamma_i^{[4n]} - \Gamma_p^{[4n]} \\ \Gamma_j^{[4n]} - \Gamma_q^{[4n]} \end{bmatrix} & \\ - \dim \mathcal{M}_{i,j,p,q} & \end{aligned}$$

for all $i, j, p, q \in \mathcal{P}$ with $i \neq j$, $p \neq q$ and $(i, j) \neq (p, q)$. Here $\mathcal{M}_{i,j,p,q}$ is given by

$$\mathcal{M}_{i,j,p,q} = \begin{cases} \mathcal{V}_i^* \cap \ker E_j \cap \ker E_q, & \text{for } i = p, \\ \{0\}, & \text{else.} \end{cases}$$

V. CONCLUSION

We have characterized σ -observability (or, equivalently, mode detectability) via rank conditions in terms of certain Kalman and Hankel matrices. This characterization utilizes a rank condition for ui-observability of an augmented system. We also introduced the weaker notion of σ_1 -observability (switch observability) which does not require individual modes to be observable and also provide a characterization based on rank criteria of certain matrices. For the weakest observability notion, t_S -observability (relevant for fault detection), we also provide a matrix-rank-based characterization.

Additionally, if the switched system is not σ -observable, we have given the precise set of initial values and inputs which produce indistinguishable outputs.

Under the strict properness assumption (A3), our result is a nice combination of the recent results on inhomogeneous switched ODEs [13] and homogeneous switched DAEs [8]; avoiding this assumption is a topic of ongoing research. Another line of research is the construction of an observer for switch observable inhomogeneous switched DAEs; we expect that the observer designs from [20] and [8] can be combined; at least with Assumption (A3) in place.

APPENDIX

Unknown-input-observability, weakly unobservable states

Unknown-input-observability for unswitched DAEs is a straightforward generalization from the ODE-case [19], [21].

Definition 10: A linear DAE

$$\Sigma : \quad E\dot{x} = Ax + Bu, \quad y = Cx \quad (16)$$

is called *unknown-input-observable (ui-observable)* if and only if $y \equiv 0$ implies $x \equiv 0$ (independently of u).

Note that in [19], [21] ui-observability is actually called strong observability; however, in the context of DAEs the notion of strong observability is used for a different concept, see e.g. the survey [22]. Therefore we use the less ambiguous notion ui-observability as, e.g., in [23].

Lemma 11: The regular system (16) with $B^{\text{imp}} = 0$ is ui-observable if, and only if,

$$\text{rk} \begin{bmatrix} \mathcal{O}_{(A^{\text{diff}}, C^{\text{diff}})}^{[n]} & \Gamma_{(A^{\text{diff}}, B^{\text{diff}}, c^{\text{diff}})}^{[n]} \end{bmatrix} = \dim \mathcal{V}^* + \text{rk}(\Gamma^{[n]}).$$

Another characterization of ui-observability is given by the absence of zeros of (16), where $\lambda \in \mathbb{C}$ is called a zero of (16) if, and only if,

$$\text{rk} \begin{bmatrix} \lambda E - A & -B \\ C & 0 \end{bmatrix} < n + \text{rk} \begin{bmatrix} -B \\ 0 \end{bmatrix}. \quad (17)$$

Lemma 12 (cf. [24] for ODE case): Consider the regular DAE (16) with $B^{\text{imp}} = 0$ and let $\mathcal{Z}(\Sigma)$ be the set of its zeros. Then (16) is ui-observable if, and only if, $\mathcal{Z}(\Sigma) = \emptyset$.

For a DAE which is not ui-observable, the input and initial conditions that lead to outputs being zero are characterized by the following result.

Lemma 13 (cf. [19] for ODE case): Consider the regular DAE (16) with $B^{\text{imp}} = 0$ and for $\lambda \in \mathbb{C}$ let $\hat{x}_0 \in \mathbb{C}^n$ and $u_0 \in \mathbb{C}^m$ be such that

$$\begin{bmatrix} \lambda E - A & -B \\ C & 0 \end{bmatrix} \begin{pmatrix} \hat{x}_0 \\ u_0 \end{pmatrix} = 0.$$

Then, for initial condition $x(0) = \text{Re } \hat{x}_0$, the output of (16) is identically zero for input $u(t) = e^{\text{Re } \lambda t} (\cos(\text{Im } \lambda t) \text{Re } u_0 - \sin(\text{Im } \lambda t) \text{Im } u_0)$.

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