Modeling water hammers via PDEs and switched DAEs with numerical justification

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Abstract: In water distribution networks instantaneous changes in valve and pump settings may introduce jumps and peaks in the pressure. In particular, a well known phenomenon in response to the sudden closing of a valve is the so called water hammer, which (if not taken into account properly) may destroy parts of the water network. It is classically modeled as a system of hyperbolic partial differential equations (PDEs). After discussing this PDE model we propose a simplified model using switched differential-algebraic equations (DAEs). Switched DAEs are known to be able to produce infinite peaks in response to sudden structural changes. These peaks (in the mathematical form of Dirac impulses) can easily be predicted and may allow for a simpler analysis of complex water networks in the future. As a first step toward that goal, we verify the novel modeling approach by comparing these two modeling techniques numerically for a simple setup consisting of two reservoirs, a pipe and a valve.

Keywords: water distribution networks; water hammer; compressible flow; switched DAEs; Dirac impulses;

1 Introduction

The occurrence of hydraulic transients in the operation of water distribution network is inevitable. Such transients are planned or accidental changes of the network configuration. These sudden structural changes can have dramatic effects in flow regimes, ranging from pump defects to catastrophic pipeline failures. The flow of water in pipes is usually modeled as system of nonlinear hyperbolic balance laws (i.e. partial differential equations, PDEs), see e.g. Izquierdo et al. (2004), where the sudden structural changes lead to large peaks and fast transients in the solution.

We propose to model such fast transients in the framework of switched differential algebraic equation (switched DAEs). This framework was originally introduced for modeling electrical circuits (Trenn, 2012) and allows a precise mathematical description of peaks and fast transients in the form of Dirac impulses and jumps.

Our focus in this paper is on the so-called water hammer, which results from sudden changes of velocity in pipelines and can cause large pressures magnitudes. It is usually created by rapidly closing valves, shutting off or restarting pumps. Our goal is to show that these pressure peaks occurring in the PDE simulations can be well approximated by a suitable switched DAE model.

The paper is organized as follows. In Section 2 the water network and its components are defined as a graph and the mathematical models of the pipes and other components (like reservoir and valves) are introduced. Section 3 provides a brief introduction to the some mathematical preliminaries for system of conservations laws and switched DAEs. In Section 4 we study in detail a simple water network which exhibits a water hammer; in particular, we derive the corresponding PDE model as well as a switched DAE model. In Section 5 a numerical comparison of the PDE and switched DAE model are presented.

2 Water network modeling

The structure of a water network can be modeled as a finite undirected graph $(V,E)$ where $V$ is the set of nodes and $E \subseteq V \times V$ is the set of edges. Each edge $e \in E$ corresponds to a pipe of the water network and the nodes $v \in V$ are the connections or endpoints of pipes, including junctions, pumps, valves, or reservoirs.

In the following we will describe the model of some water network element. The two main physical quantities involved in the descriptions are pressure and flow, whose values at the end points of the pipes are related to each other corresponding to the type of node. Furthermore, the modeling of the flow in the pipes also involves density of the water. Usually, water is assumed to be incompressible, i.e. the density is assumed to be constant. However, our focus is on modeling the water hammer effect and for this it is necessary to take into account the (slight) compressibility of water.

2.1 Models of water flow in pipe

The water flow in a pipe can be modeled in two different ways depending on whether the compressibility of water is taken into account or not. In order to study transient phenomena like water hammer it is necessary to model compressibility, in particular, density and mass flow become space-dependent quantities. On the other hand, to understand the qualitative behavior, in particular, in large networks, it often suffices to model water as incompressible fluid. We will briefly introduce both models in the following.

2.1.1 Compressible flow in a pipe

Following Wylie and Streeter (1978) and Adami et al. (2012) we use the following pressure law for compressible fluids:

$$P(\rho) = P_a + K \frac{\rho - \rho_a}{\rho_a},$$ (1)
where $K > 0$ is the so-called bulk modulus, $P_a > 0$ is the atmospheric pressure and $\rho_a > 0$ is the density at atmospheric pressure. The bulk modulus is related to the speed of sound $c > 0$ as follows:

$$c^2 = \frac{\partial P}{\partial \rho} = K/\rho_a.$$  

(2)

Note that $\beta := 1/K$ is the so-called compressibility coefficient.

We consider a completely filled pipe of length $L > 0$ with mass density $\rho(x,t) > 0$ and mass flux $q(x,t) \in \mathbb{R}$ both defined on $[0,L] \times \mathbb{R}_+$. The compressible flow of water in the pipe can be modelled by the balance law of the following form (Herty et al., 2010, Sec. 2):

$$\partial_t \rho + \partial_x q = 0,$$

$$\partial_t q + \partial_x \left( \frac{q^2}{\rho} + P(\rho) \right) = -c_f \frac{\partial q}{\partial x},$$

(3)

with the pressure law $P : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ given by (1) and where $c_f > 0$ is the friction against the pipe wall and $D > 0$ is the diameter of the pipe. The initial condition for (3) is:

$$q(x,0) = q_0(x) \text{ and } P(\rho(x,0)) = \rho_0(x) \quad x \in [0,L],$$

(4)

for some initial flow $q_0 : [0,L] \rightarrow \mathbb{R}$ and some initial pressure $\rho_0 : [0,L] \rightarrow \mathbb{R}_+$. Note that the initial condition is given implicitly in terms of the pressure and not explicitly in terms of the density. The reason is that the pressure is the more relevant physical quantity, in particular, when the pipes are coupled with other water network elements. When the individual pipes are connected with other elements of the overall water distribution network, additional boundary conditions will be imposed.

### 2.1.2 Quasi-stationary water flow model

After some initial transient behaviour, the water flow in the pipe may be assumed to get stationary, i.e. the flow is location independent and we write $Q(t) = \frac{dQ}{dt}$ (mass flux is mass flow per unit area), $x \in [0,L]$ and where $A = \pi D^2/4$ is the area of the pipe. Furthermore the density is assumed constant in space and time, i.e. $\rho(x,t) = \rho$ for $(x,t) \in [0,L] \times \mathbb{R}_+$ and the pressure variable $p(x,t)$ is not coupled to the density via (1) anymore (in particular, water is considered incompressible). The remaining dynamical behavior in the variables $Q(t), P_0(t) = P_0(t) = (0,0)$ and $P_2(t) = P_2(t)$, $t \in (0,\infty)$ can be described by the following ODE (Jansen and Tischendorf, 2014; Chaudhry and Mays, 2012):

$$\frac{dQ}{dt} + \frac{A}{L} (P_L - P_0) - c_f \frac{Q}{2DA \rho} = 0.$$

(5)

### 2.2 Other network elements

#### 2.2.1 Reservoir

A reservoir is a node in the water network graph with arbitrary mass flow but with given pressure. In particular, connecting a pipe with its left side to a reservoir adds a boundary condition for $p(0,t)$ in the PDE model (3) and constraints the variable $P_0$ in the ODE-model (5).

Fig. 1. Valve in combination with reservoir

#### 2.2.2 Valve

For simplicity, a valve here is modeled in combination with a reservoir, see Figure 1, which imposes the following algebraic constraints.

- valve open: $P_V = P_D$;
- valve closed: $Q_V = 0$,

where $P_V, Q_V$ denotes the pressure and flow through the valve and $P_D$ is the reservoir pressure. In the next section we present some general mathematical preliminaries related to (3) and (5).

### 3 Preliminaries

#### 3.1 Systems of hyperbolic balance laws

We consider systems of balance laws in one space dimension of the following form

$$U_t + (F(U))_x = S(U), \quad \text{on } [a,b] \times \mathbb{R}_+,$$

$$U(x,0) = U_0(x), \quad x \in \Omega,$$

$$\Psi_1(t, U(t,a)) = 0, \quad t > 0,$$

$$\Psi_2(t, U(t,b)) = 0, \quad t > 0,$$

(6a)

(6b)

(6c)

(6d)

where $U : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$ with $\Omega := [a,b] \subseteq \mathbb{R}^n$ being the domain and the open connected set $U \subseteq \mathbb{R}^n$ being the range of the problem, $F : U \rightarrow \mathbb{R}^n$ is the flux function, $S : U \rightarrow \mathbb{R}^n$ is the source term, $U_0 : \Omega \rightarrow \mathbb{R}$ is the initial data and $\Psi_1 : \mathbb{R}_+ \times U \rightarrow \mathbb{R}^n$, $i = 1,2$, $b_1 + b_2 = n$, are the time-varying, implicit boundary conditions. If the flux $F$ is differentiable, then the PDE (6a) can be written in quasi-linear form by

$$U_t + A(U)U_x = S(U),$$

(7)

where $A(u) := DF(u)$ for $u \in U$ and $DF$ denotes the Jacobian of $F$.

The system (6) does not necessarily admit a classical (i.e. differentiable) solution, even for “well-behaved” initial data $U_0$. Hence, weak solutions will be considered defined as follows (c.f. Bressan (2013)).

Definition 1. (Weak solution). A function $U : \Omega \times \mathbb{R}_+ \rightarrow U$ is a weak solution of (6a) if, and only if, $U, F(U)$ and $S(U)$ are locally integrable and for every $\phi \in C^1(\mathbb{R}_+) \rightarrow \mathbb{R}$ with compact support, one has

$$\int_{\mathbb{R}_+ \times \Omega} (U\phi_t + F(U)\phi_x) dxdt = -\int_{\mathbb{R}_+ \times \Omega} S(U)\phi dxdt.$$  

(8)

The boundary values are assumed to be satisfied in the Godunov/trace sense, i.e. see [Colombo and Garavello (2008), Colombo and Garavello (2006)] for more details.

When considering weak solutions of (6) uniqueness of solutions cannot be expected in general. Under certain assumptions, uniqueness can be recovered by imposing a so called entropy condition as follows (c.f. Freistuhler (1998)).

Definition 2. (Entropy solution). A differentiable function $\eta \in C^1(\mathbb{R}_+ \rightarrow \mathbb{R})$ is called an entropy solution of the PDE (6a) if there exists an entropy flux $q \in C^1(\mathbb{R}_+ \rightarrow \mathbb{R})$ such that

$$Dq(\eta(u)) \cdot DF(u) = Dq(\eta(u)) \quad \forall u \in U.$$  

(9)

For a given entropy $\eta$ with corresponding entropy flux $q$, a weak solution $U$ of (6a) is called entropy solution if it additionally satisfies

$$\eta(U)_t + q(U)_x \leq Dq(\eta(u)) \cdot S(U)$$

in a weak sense.

Remark 3. For any classical (i.e. differentiable) solution $U$ of (6a) it is easily seen that (9) implies

$$\eta(U)_t + q(U)_x = Dq(\eta(u)) \cdot S(U).$$

For existence and uniqueness of weak entropy solutions of the initial/boundary value problem (6) the following well-posedness assumptions are usually imposed (c.f. Borsche et al. (2012)):

- **(B-I) Bounded variation**
  The initial data $U_0$ is a function of bounded variation with sufficiently small total variation.

- **(B-II) Strict hyperbolicity**
  The system of balance laws (6a) in quasi-linear form (7) is strictly hyperbolic, i.e., for every $u \in U \subseteq \mathbb{R}^n$, the Jacobian matrix $A(u)$ of the flux function $F$
has \( n \) real, distinct eigenvalues denoted by \( \lambda_i(u) \), \( i = 1, \ldots, n \) and are ordered as follows:
\[
\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u).
\]

**Proposition 6.** A matrix pair \((E, A) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}\) is regular if and only if there exist invertible transformation matrices \( S, T \in \mathbb{R}^{n \times n} \) which put \((E, A)\) into QWF [QWF, c.f. Weierstraß (1868)]
\[
(SET, SAT) = \left( \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right)
\]
where \( N \in \mathbb{R}^{n_2 \times n_2} \), with \( 0 \leq n_2 \leq n \) is a nilpotent matrix, \( J \in \mathbb{R}^{n_1 \times n_1} \) with \( n_1 := n - n_2 \) is some matrix and \( J \) is the identity matrix of the appropriate size. Moreover, index of the linear DAE is the minimum value \( n' \) for which \( N^{n'} = 0 \). The notion of the index of a DAE can be generalized to nonlinear DAEs in the form of the differential index see Borutzky (2009) for details.

**4 Analysis of a simple water network**

We want to study a simple water network consisting of an upstream reservoirs with given pressure \( P_U \), one pipe of length \( L \) and a valve combined with a downstream reservoir with pressure \( P_D < P_U \), see Figure 2. This setup is used in Chen et al. (2015), to study optimal boundary condition to mitigate water hammer, and in Szydlowski (2002) for the construction of accurate finite volume scheme. Here, this setup is used to compare the novel modeling approach via switched DAEs with the more classical PDE models with regard to the water hammer effect.

![Fig. 2. Simple set up for water hammer](image)

It is well known that such a configuration will result in a water hammer if the valve is instantaneously closed. Our goal is to compare the PDE model and the switched DAE model for this configuration with respect to capturing the water hammer effect. For that we will first discuss the solution concepts for both approaches for the specific example water network.

**4.1 PDE solution framework**

The balance law in the pipe is given by (3), i.e. in terms of (6a) we have
\[
\Omega = [0, L], \quad \mathcal{U} \subset \mathbb{R}^+ \times \mathbb{R}, \quad U(x, t) = \left( \begin{array}{c} \rho(x, t) \\ \rho u(x, t) \end{array} \right),
\]
and, for \( u = (\rho, q) \in \mathcal{U} \),
\[
F(u) = \left( \begin{array}{c} q \\ \rho u + q u \end{array} \right), \quad S(u) = \left( \begin{array}{c} 0 \\ -e_j \end{array} \right),
\]
with pressure law (1). The initial condition is given by (4) where the initial density \( \rho_0 \) is induced by the usually considered initial pressure \( p_0 \) via the invertible pressure law. The boundary conditions in terms of (6c),(6d) are
\[
\Psi_1(t, (\rho, q)) = P(\rho) - P_U, \quad \forall t \in \mathbb{R}^+,
\]
and
\[
\Psi_2(t, (\rho, q)) = \left\{ \begin{array}{ll} P(\rho) - P_D, & t \in [0, t_s), \\ q, & t > t_s. \end{array} \right.
\]

Note that the discontinuity induced by the switch only occurs in the boundary condition, therefore wellposedness can be studied for each mode individually as the PDE can be “restated” at time \( t = t_s \) with the initial value given by the final value of the solution on the time interval \([0, t_s)\); in particular, the wellposedness conditions (B-II), (B-III), (B-IV) can be check independently of the valve's state.
The quasi-stationary model (5) together with the valve-

\[ A(\rho, q) = \begin{pmatrix} 0 & -\rho \frac{\partial g}{\partial \rho} \\ \frac{1}{\rho} & \rho \frac{\partial g}{\partial \rho} \end{pmatrix} \]

and invoking the pressure law (1) we see that \( P'(\rho) = \frac{K}{\rho_a} > 0 \) independently of \( \rho \). Hence the eigenvalues of \( A(\rho, q) \) are

\[ \lambda_{1/2} = \frac{q}{\rho} \pm \sqrt{\frac{K}{\rho_a}}. \]

Consequently, \( \lambda_1(\rho) > \lambda_2(\rho) \) for all \( u, (\rho, q) \in \mathcal{U} \) and hence (3) with pressure law (1) is strictly hyperbolic.

The eigenvectors of \( A(u) \) are

\[ r_1(u) = (\lambda_1, 1) \quad \quad r_2(u) = (\lambda_2, 1). \]

Hence

\[ D\lambda_1(u) \cdot r_1(u) = \left[ -\frac{q}{\rho} + \frac{1}{\rho} \right] \left( \frac{1}{2} + \sqrt{\frac{2}{\rho}} \right) \]

and

\[ D\lambda_2(u) \cdot r_2(u) = -\frac{1}{\rho} \sqrt{\frac{K}{\rho_a}} \quad \forall u, (\rho, q) \in \mathcal{U}. \]

Consequently, genuine nonlinearity is established.

Clearly \( S \) is locally Lipschitz continuous.

In order to have sign-definite eigenvalues, i.e.

\[ \lambda_1(\rho) > 0 > \lambda_2(\rho) \quad \forall u \in \mathcal{U}, \]

we have to restrict our attention to the so-called subsonic case, i.e.

\[ \mathcal{U} \subseteq \left\{ (\rho, q) \in \mathbb{R}_+ \times \mathbb{R} \mid \frac{q}{\rho} < \sqrt{\frac{K}{\rho_a}} \right\}. \]

In that case we have \( R^+(u) = [r_1(u)] \) and

\[ D_u \Psi_1(t, u) \cdot R^+(u) = [P'(\rho)] \left[ \lambda_1, 1 \right] = P'(\rho) \neq 0, \]

i.e. condition (10a) is satisfied for all \( u, (\rho, q) \in \mathcal{U} \) and \( t > 0 \).

For the right boundary consider \( R^-(\mathbf{n}) = [r_2(\mathbf{n})] \)

\[ D_e \Psi_2(t, \mathbf{n}) \cdot R^-(\mathbf{n}) = \left[ \begin{array}{c} [P'(\mathbf{n})] \\ \lambda_2(\mathbf{n}) \end{array} \right] = P'(\mathbf{n}) \neq 0, \quad t \in (0, t_S), \]

and

\[ \mathbf{n} = (\mathbf{n}, g) \in \mathcal{U}; \]

hence, due to the restriction to the subsonic case, condition (10b) is satisfied.

Altogether, we can expect numerical simulations to result in reasonable approximations of solutions of the corresponding initial/boundary value problem (6) as long as the solution remain in the subsonic case.

### 4.2 Switched DAE framework

The quasi-stationary model (5) together with the valve-dependence of the pressure condition (valve open on \( [0, t_S] \) and valve closed on \( [t_S, \infty) \)) for a setup as shown in Figure 2 leads to a switched DAE of the form

\[ E_\sigma \dot{x} = A_\sigma x + f + g(x), \quad \sigma(t) = \begin{cases} 1, & t \in [0, t_S), \\ 2, & t \geq t_S, \end{cases} \]

where \( x = (Q, P_0, P_L)^T \), \( \sigma(t) = \begin{cases} 1, & t \in [0, t_S), \\ 2, & t \geq t_S, \end{cases} \)

\[ E_1 = E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & c_1 & -c_1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & c_1 & -c_1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} (0, -P_0, -P_L)^T, \\ (0, -P_0, 0)^T, \\ (0, 0, 0)^T \end{bmatrix}, \quad g_1(x) = g_2(x) \]

\[ = g(x) = \left( \begin{array}{c} e_2^T Q |Q| \\ 0 \end{array} \right), \quad c_1 = \frac{A}{L} > 0, c_2 = \frac{c_f}{2D A p_0} > 0 \]

(14)

Note that the switched DAE (13) contains a nonlinear term \( g_0(x) \) and therefore the distributional solution framework recalled in Section 3.2 cannot be applied directly. Nonlinear switched DAEs were investigated in Liberzon and Trenn (2012), but this approach excludes Dirac impulses in \( x \) by definition, because if a Dirac impulse occurs in the solution of \( x \) of (13) (which we actually desire to capture the water hammer effect) then it is unclear how \( g_0(x) \) has to be evaluated in general (e.g. what is the sine of a Dirac impulse). However, for our specific case we see that the nonlinear term \( g(x) = g_1(x) = g_2(x) \) can be written as follows:

\[ g(x) = N g(M x) \]

where

\[ M = [1 \ 0 \ 0], \quad N = M^T, \quad g(Q) = -c_2 Q |Q|. \]

This special structure of the nonlinearity \( g \) allows us to extend the distributional theory from the linear case also to the nonlinear case. In particular, we have a well defined solution concept:

**Definition 7.** Consider a general nonlinear switched DAE of the form (13) with \( E_i, A_i \in \mathbb{R}^{n \times n}, i \in \{1, \ldots, P\} \), and \( g_i : \mathbb{R}^n \rightarrow \mathbb{R}^n \) assume that there exists \( \mathcal{N} \in \mathbb{R}^{n \times m} \), \( \mathcal{M} \in \mathbb{R}^{k \times n} \), \( \mathcal{N} \mathcal{M} = \mathbb{R}^k \rightarrow \mathbb{R}^m \) with \( m, k \leq n \) such that

\[ g_i(\xi) = \mathcal{N} \mathcal{M} \xi \quad \forall \xi \in \mathbb{R}^n. \]

Then any pair \((x, f)\) in \((\mathbb{D} \mathbb{p} \mathbb{c} \mathbb{w})^{n_1} \times (\mathbb{D} \mathbb{p} \mathbb{c} \mathbb{w})^{n_2} \) is a solution of (13) if,

1. \( \mathcal{M} x \) is impulse-free, i.e. there exists a piecewise-smooth function \( \mathcal{F} : \mathbb{R} \rightarrow \mathbb{R}^m \) such that \( \mathcal{M} x \) is the distribution induced by the function \( \mathcal{F} \),
2. \( \mathcal{N} \mathcal{M} \mathcal{F} \) is a piecewise-smooth function, and
3. \( E_i x = A_i x + f + G x \) holds as an equality within the space of piecewise-smooth distributions where \( G \) is the distribution induced by the piecewise-smooth function \( \mathcal{N} \mathcal{M} \mathcal{F} \).

Characterizing existence and uniqueness of solution in the sense of Definition 7 for general nonlinear switched DAEs (13) is still an open problem and outside the scope of this contribution. However, it is possible to derive an existence and uniqueness result for the specific case considered here:

**Lemma 8.** Consider the nonlinear initial-boundary problem (ITP)

\[ E(\sigma)(x) = x(0, 0, 0) \]

(15)

\[ (E(\sigma)(x))_{0, \infty} = (A \dot{x} + f + g(x))_{0, \infty} \]

where either \((E, A) = (E_1, A_1)\) or \((E, A) = (E_2, A_2)\) and \( g(x) = g_1(x) = g_2(x) \) as in (14). Then for every initial trajectory \( x_0 \in (\mathbb{D} \mathbb{p} \mathbb{c} \mathbb{w})^{n_1} \) and every inhomogeneity \( f \) induced by a piecewise-smooth function, there exists a unique solution \( x \in (\mathbb{D} \mathbb{p} \mathbb{c} \mathbb{w})^{n_1} \) of the ITP in the sense of Definition 7.

**Proof.** Case \((E, A) = (E_1, A_1)\).

The index 1 DAE \( (E(\sigma) x) = f + g(x) \) with \( x = (Q, P_0, P_L)^T \) and \( f = (f_1, f_2, f_3)^T \) reads as

\[ \dot{Q} = c_1 (P_0 - P_L) + f_1 - c_2 Q + \mathcal{F}, \]

\[ 0 = f_1 + f_2, \]

\[ 0 = f_1 + f_3. \]
which is equivalent to \( P_0 = -f_2, P_L = -f_3 \) and
\[
\dot{Q} = c_1(f_3 - f_2) + f_1 - c_2 Q|Q|.
\] (15)
The latter is an ODE where existence and uniqueness of (local) solutions is guaranteed. Although the right-hand side is not globally Lipschitz-continuous we nevertheless can conclude global existence of solutions, because of the negative sign in the quadratic term. The corresponding ITP inherits the existence and uniqueness result, because inconsistent initial values in \( P_0 \) and \( P_L \) just lead to simple jumps.

Case (E, A) = (E_2, A_2).

The index 2 DAE \( \dot{x} = Ax + f + g(x) \) with \( x = (Q, P_0, P_L)' \) and \( f = (f_1, f_2, f_3)' \) reads as
\[
\dot{Q} = c_1(P_0 - P_L) + f_1 - c_2 Q|Q|
\]
\[
0 = P_0 + f_2
\]
\[
0 = Q + f_3
\]
For the corresponding ITP with initial trajectory \( x^0 = (Q^0, P_0^0, P_L^0) \) we can directly derive the following unique solution:
\[
Q = Q^0\left( -\infty, 0 \right) - f_3|\left( -\infty, 0 \right)|
\]
\[
P_0 = P_0^0\left( -\infty, 0 \right) - f_2|\left( -\infty, 0 \right)|
\]
\[
P_L = P_L^0\left( -\infty, 0 \right) + \frac{1}{c_i} (\dot{Q} + c_1 P_0 + f_1 - c_2 Q|Q|)\left( 0, \infty \right)
\]
\[
= P_0^0\left( -\infty, 0 \right) + \frac{1}{c_i} Q^0(0^{-})\delta_0
\]
\[
+ \frac{1}{c_i} \left( f_3|\left( 0, \infty \right) + (-f_2 + \frac{f_1}{c_1}f_3 + \frac{1}{c_2} f_3|f_3|)\left( 0, \infty \right),\right)
\]
where \( \delta_0 \) denotes the Dirac impulse located at \( t = 0 \).

Corollary 9. The switched nonlinear DAE (13), (14) has for every initial condition \( Q(0) = Q^0 \in \mathbb{R} \) a unique solution in the sense of Definition 7. In particular, the jump and the Dirac impulse in \( P_L(t) \) at \( t_s \) are given by:
\[
P_L(t_s) = P_U, \quad P_L(t_s) = \frac{1}{c_1} Q(t_s) \delta_{t_s}.
\]

Comparison of both modeling approaches

Our longterm goal is to rigorously prove that the solution \( (q, P(t)) \) of the balance law (6) given by (3), (1), (4), (12) converges in an appropriate sense to the solutions \( (Q, P_0, P_L) \) of the switched DAE (13), (14) as the compressibility coefficient \( \beta \) approaches zero. Here we will focus on the jump and the Dirac impulse in the pressure due to closing the valve. In particular, we assume that the initial condition (4) is such that the PDE solution on \( [0, t_s] \) is stationary, i.e. \( q(t, x) \) is constant in time and space (or in other words, when the valve is closed the dynamics in the pipe have settled down). For the numerical simulations we use a HEEOC type AEDM scheme of order 3 (Borsche and Kall, 2014). For the detailed characteristics analysis see [chapter 5, Kall (2016)]. The upper part of Figure 3 shows the results for the pressure value at the valve over the time interval \([0.4s, 4s]\) with initial values
\[
q_0(x) \equiv 0, \quad \rho_0(x) \equiv 1.4115 \times 10^3
\]
and pipe parameters
\[
P_a = 1.01 \times 10^6, \quad \beta = \frac{1}{K} = 4 \times 10^{-9}, \quad \rho_a = 1000, \quad L = 5, \quad D = 0.5, \quad c_f = 0.02.
\]
Clearly, there is a strong pressure spike just after the switching time \( t_s = 0.5s \) and then the pressure periodically settles to a new pressure value. The frequency of this periodic behavior is determined by the pipe length \( L \) (the larger \( L \) the lower the frequency) and the speed of sound for higher compressibility coefficients \( \beta \).

In order to compare both results, we first have to obtain an approximation of the pressure value \( p(t, L) \) as \( t \) tends to infinity. Instead of running the simulation for a very long time, we just chose a settling time \( \varepsilon > 0 \) and take the average of \( p(t, L) \) on the interval \([t_s + \varepsilon, T] \) where \( T > t_s + \varepsilon \) is our overall simulation time, i.e.
\[
\overline{p}_L(t_s) := \frac{1}{T - (t_s + \varepsilon)} \int_{t_s + \varepsilon}^{T} p(t, x) \, dt.
\]
with
\[
\varepsilon = 1.5, \quad T = 4.
\]
we obtain
\[
\overline{P}_L \approx 8.23 \times 10^8
\]
The value predicted by the switched DAE is
\[
P_L(t_s) = P_U \approx 8.23 \times 10^8
\]
In Table 1 the relative error between \( \overline{P}_L \) and \( P_L(t_s) \) is presented for decreasing compressibility coefficients \( \beta \).

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( \overline{P}_L - P_L(t_s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.0 \times 10^{-9}</td>
<td>8.2336 \times 10^8</td>
</tr>
<tr>
<td>2.0 \times 10^{-9}</td>
<td>8.3329 \times 10^8</td>
</tr>
<tr>
<td>5.0 \times 10^{-10}</td>
<td>8.2305 \times 10^8</td>
</tr>
<tr>
<td>2.5 \times 10^{-10}</td>
<td>8.2303 \times 10^8</td>
</tr>
<tr>
<td>1.25 \times 10^{-10}</td>
<td>8.2299 \times 10^8</td>
</tr>
</tbody>
</table>

Table 1. Pressure at valve comparison for long after switching.

Already for the largest value of \( \beta \), the value \( P_L(t_s) \) is a very good approximation of \( \overline{P}_L \) and the approximation gets better for decreasing \( \beta \).

In order to compare the peak in \( p(t, L) \) just after the valve is closed with the Dirac impulse \( P_L(t_s) \) in response to the switching time, we recall that a Dirac impulse \( \delta_{t_s} \) at \( t_s > 0 \) can be approximated by a sequence of functions \( t \mapsto \delta_{t_s}(t) \) such that \( \delta_{t_s}(t) = 0 \) for \( t \neq [t_s, t_s + \varepsilon] \) and \( \int_{t_s + \varepsilon}^{t_s + \varepsilon} \delta_{t_s}(t) \, dt = 1 \). We therefore make the Ansatz
\[
p(t, L) \approx \overline{P}_L^{\text{imp}} \delta_{t_s + \varepsilon}(t) + \overline{P}_L, \quad t \in (t_s, T],
\]
and hence, we can approximate the magnitude of the “smoothed-out” Dirac impulse occurring in the PDE model as follows:
\[
\overline{P}_L^{\text{imp}} := \int_{t_s + \varepsilon}^{t_s + \varepsilon} p(t, L) - \overline{P}_L \, dt.
\]
The Dirac impulse induced by the switched DAE is (see Corollary 9):
\[
P_L(t_s) = \frac{1}{c_1} Q(t_s) \delta_{t_s} =: P_L^{\text{imp}} \delta_{t_s}.
\]
Similar as for the PDE simulations we assume that the DAE is stationary before we switch, i.e. \( Q(t_s) \) is obtained by solving
\[
0 = \dot{Q} = c_1 (P_U - P_D) - c_2 Q|Q|
\]
With the above chosen parameters we get:
\[
Q(t_s) = 1.2830 \times 10^6, \quad P_L^{\text{imp}} = \frac{1}{c_1} Q(t_s) = 3.2670 \times 10^7.
\]
A comparison between \( P_L^{\text{imp}} \) and \( P_L^{\text{imp}} \) for different values of the compressibility coefficient \( \beta \) is presented in Table 2.

For large \( \beta \) the approximation is not very accurate, however, for decreasing compressibility the accuracy of the approximation drastically improves. Finally, we want to stress that the choice of \( \varepsilon \) influences the approximation accuracy, see Table 3.

However, the qualitative behavior of a decreasing error for decreasing compressibility coefficient remains valid.
Fig. 3. Comparison of pressure profile at valve ($P_L$) with PDE models (above) and switched DAE model (below)

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\tau_{tg}^{imp}$</th>
<th>$P_L^{imp}$</th>
<th>$RE := \frac{P_L^{imp} - P_L^{imp}<em>{tg}}{P_L^{imp}</em>{tg}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4.0 \cdot 10^{-9}$</td>
<td>4.1251 $\cdot 10^4$</td>
<td>3.2670 $\cdot 10^7$</td>
<td>0.2626</td>
</tr>
<tr>
<td>$2.0 \cdot 10^{-9}$</td>
<td>3.4758 $\cdot 10^5$</td>
<td>3.2670 $\cdot 10^7$</td>
<td>0.0639</td>
</tr>
<tr>
<td>$5.0 \cdot 10^{-10}$</td>
<td>3.1817 $\cdot 10^7$</td>
<td>3.2670 $\cdot 10^7$</td>
<td>0.0261</td>
</tr>
<tr>
<td>$2.5 \cdot 10^{-10}$</td>
<td>3.2989 $\cdot 10^7$</td>
<td>3.2670 $\cdot 10^7$</td>
<td>0.0184</td>
</tr>
<tr>
<td>$1.25 \cdot 10^{-10}$</td>
<td>3.2989 $\cdot 10^7$</td>
<td>3.2670 $\cdot 10^7$</td>
<td>0.0083</td>
</tr>
</tbody>
</table>

Table 2. Impulse length comparison

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\epsilon = 1$</th>
<th>$\epsilon = 1.5$</th>
<th>$\epsilon = 2$</th>
<th>$\epsilon = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4.0 \cdot 10^{-9}$</td>
<td>0.1812</td>
<td>0.2626</td>
<td>0.2616</td>
<td>0.2697</td>
</tr>
<tr>
<td>$2.0 \cdot 10^{-9}$</td>
<td>0.0508</td>
<td>0.0639</td>
<td>0.0809</td>
<td>0.0808</td>
</tr>
<tr>
<td>$5.0 \cdot 10^{-10}$</td>
<td>0.0438</td>
<td>0.0261</td>
<td>0.0253</td>
<td>0.0233</td>
</tr>
<tr>
<td>$2.5 \cdot 10^{-10}$</td>
<td>0.0226</td>
<td>0.0184</td>
<td>0.0016</td>
<td>0.0160</td>
</tr>
<tr>
<td>$1.25 \cdot 10^{-10}$</td>
<td>0.0084</td>
<td>0.0083</td>
<td>0.0078</td>
<td>0.0053</td>
</tr>
</tbody>
</table>

Table 3. Error comparison with different $\epsilon$

6 Conclusion

We have presented a switched DAEs model for water hammer on a simple setup, which we compared with a compressible nonlinear system of balance laws. With the support of numerical simulations of the PDE model we justified our conjecture that a switched DAE model is a good approximation for the PDE model with small compressibility coefficient. Future work will focus on a formal proof of convergence as well as the treatment of larger networks.

References


