Tracking control: performance funnels and prescribed transient behaviour

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Abstract

Tracking of a reference signal (assumed bounded with essentially bounded derivative) is considered in a context of a class of nonlinear systems, with output y, described by functional differential equations (a generalization of the class of linear minimum-phase systems with positive high-frequency gain). The primary control objective is tracking with prescribed accuracy: given $\lambda > 0$ (arbitrarily small), determine a feedback strategy which ensures that, for every admissible system and reference signal, the tracking error $e = y - r$ is ultimately smaller than λ (that is, $\|e(t)\| < \lambda$ for all t sufficiently large). The second objective is guaranteed transient performance: the evolution of the tracking error should be contained in a prescribed performance funnel $\mathcal{F}$. Adopting the simple non-adaptive feedback control structure $u(t) = -k(t)e(t)$, it is shown that the above objectives can be attained if the gain is generated by the nonlinear, memoryless feedback $k(t) = K_{\mathcal{F}}(t, e(t))$, where $K_{\mathcal{F}}$ is any continuous function exhibiting two specific properties, the first of which ensures that, if $(t, e(t))$ approaches the funnel boundary, then the gain attains values sufficiently large to preclude boundary contact, and the second of which obviates the need for large gain values away from the funnel boundary.

Key words: Output feedback, transient behaviour, tracking, functional differential equations

1 Introduction

By way of motivation, consider the well-studied (see for example (Mareels, 1984; Morse, 1983; Willems and Byrnes, 1984)) class of finite-dimensional, real, linear, minimum-phase, M-input ($u(t)$), M-output ($y(t)$) systems having high-frequency gain $B \in \mathbb{R}^{M\times M}$ with $B + B^T > 0$. Systems of this class can, in suitable coordinates, be expressed in the form of two coupled subsystems

$$\left.\begin{aligned} \dot{y}(t) &= A_1 y(t) + A_2 z(t) + Bu(t), \; y(0) = y^0 \\ \dot{z}(t) &= A_3 y(t) + A_4 z(t), \qquad\qquad z(0) = z^0 \end{aligned}\right\} \tag{1}$$

with $y(t), u(t) \in \mathbb{R}^M$, $z(t) \in \mathbb{R}^{N-M}$, and where A_4 has spectrum in the open left half complex plane. Introducing the linear operator T given by

$$(Ty)(t) := A_1 y(t) + A_2 \int_0^t \exp(A_4(t-s)) A_3 y(s) ds \tag{2}$$

and the function p given by $p(t) := A_2 \exp(A_4 t) z^0$, then system (1) can be interpreted as

$$\dot{y}(t) = p(t) + (Ty)(t) + Bu(t), \quad y(0) = y^0. \tag{3}$$

In a precursor (Ilchmann *et al.*, 2002*b*) to the present paper, (1) formed a prototype subclass of a considerably more general class of nonlinear systems described by functional differential equations of the form

$$\dot{y}(t) = f\big(p(t), (Ty)(t), u(t)\big), \quad y_{[-h,0]} = y^0\,,$$

where, loosely speaking, the parameter $h \geq 0$ quantifies system "memory", p may be thought of as a (bounded) disturbance term, and T is a nonlinear causal operator. Whilst a full description of the system class is postponed to Section 2, we remark here that diverse phenomena are incorporated within the class including, for example, diffusion processes, delays (both point and distributed) and hysteretic effects. For this general system class, the problem of output tracking with prescribed asymptotic accuracy and prescribed transient output behaviour was formulated, in (Ilchmann *et al.*, 2002*b*), in terms of a performance funnel $\mathcal{F}$ determined by the graph of the set-valued map $t \mapsto F(t) = \{(t,e) |\, \varphi(t)\|e\| < 1\} \subset \mathbb{R}^M$ for suitably chosen φ; the goal was a control structure which, for every admissible system and reference signal, ensures that the graph of the tracking error $e(\cdot)$ is contained in $\mathcal{F}$. This goal was achieved

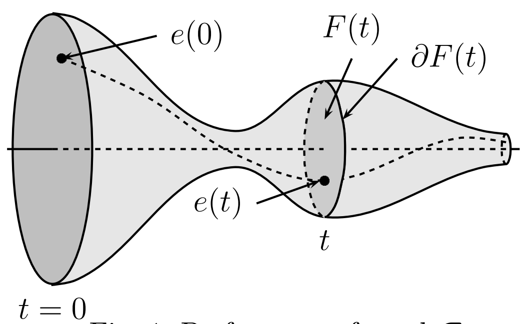

Fig. 1. Performance funnel $\mathcal{F}$.

by the simple control structure $u(t) = -k(t)e(t)$ with the gain generated by a nonlinear, memoryless feedback law of the form $k(t) = K_{\mathcal{F}}(t, e(t))$, where $K_{\mathcal{F}}$ is a continuous function such that, loosely speaking, the reciprocal $1/K_{\mathcal{F}}(t, e)$ provides a particular measure of distance of (t, e) from the boundary $\partial\mathcal{F}$ of the funnel $\mathcal{F}$ (with the effect that, if the error approaches the boundary, then the gain increases which, in conjunction with a high-gain property of the underlying system class, precludes contact with the boundary). The purpose

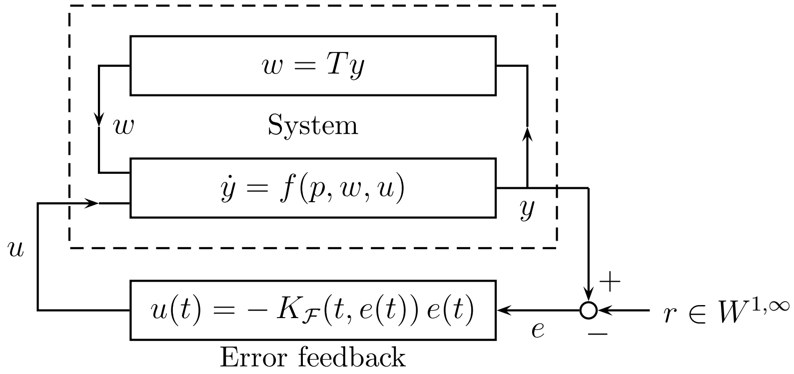

Fig. 2. Universal error feedback control.

of the present paper, *vis à vis* its precursor (Ilchmann *et al.*, 2002*b*), is to extend the choice of admissible gain functions $K_{\mathcal{F}}$, allowing for greater flexibility in the choice of measure of the distance to the funnel boundary. Colloquially speaking, the controllers in (Ilchmann *et al.*, 2002*b*) look "vertically" in the funnel in the sense that, at time t, only the instantaneous funnel information $F(t)$ is used. This approach is typified by a gain function $K_{\mathcal{F}}$ determined by the reciprocal of the vertical distance to the funnel boundary

$$K_{\mathcal{F}}(t, e) \;=\; \frac{\varphi(t)}{1 - \varphi(t)\,\|e\|} \;=\; \frac{1}{\operatorname{dist}(e, \partial F(t))}, \tag{4}$$

with the convention that, if $\varphi(t) = 0$, then $\operatorname{dist}(e, \partial F(t)) := \infty$ (in which case $K_{\mathcal{F}}(t, e) = 0$). By contrast, the present paper exploits the freedom to look also "forward" in the funnel in the sense that, at time t, the funnel information $\{F(\tau) \mid \tau \geq t\}$ is available for use. This approach has the potential to mitigate large excursions in control values by sensing, in advance, rapid changes in the funnel geometry and adjusting the control gain accordingly. The approach is typified by a gain function $K_{\mathcal{F}}$ determined by the reciprocal of the forward or future distance to the funnel

$$K_{\mathcal{F}}(t, e) \;=\; \frac{1}{d_f(t, e)}, \quad d_f(t, e) := \inf_{\tau > t} \sqrt{(\tau - t)^2 + \Big(\operatorname{dist}(e, \partial F(\tau))\Big)^2}. \tag{5}$$

Furthermore, to facilitate implementation, we also study a numerical future distance (essentially a numerical approximation to (5)).

The control strategy, investigated in (Ilchmann *et al.*, 2002*b*) and the present paper, is essentially applicable to the same system class widely studied in high-gain adaptive control. Loosely speaking, the system class encompasses

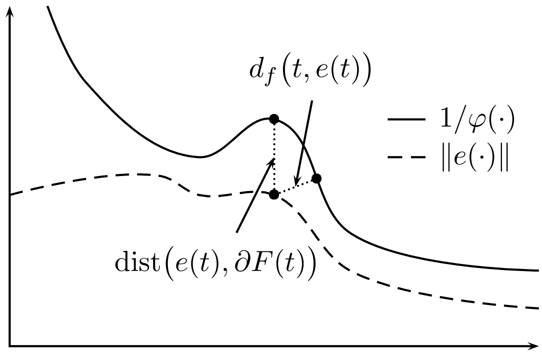

Fig. 3. The distance $d_f(t, e(t))$ to the future funnel boundary, and the vertical distance $\mathrm{dist}(e(t), \partial F(t))$ to the funnel boundary.

nonlinear counterparts of the class of linear relative degree one systems with stable zero dynamics and high-frequency gain of known sign. The main differences between the approach of the present paper (and its precursor (Ilchmann *et al.*, 2002*b*)) and adaptive control strategies in the literature (see (Ilchmann *et al.*, 2002*a*) and the reference therein) are: (i) prescribed transient behaviour is guaranteed, (ii) the gain $t \mapsto k(t)$ is not a monotonically non-decreasing function, (iii) the gain is not adaptively tuned by a dynamical system (c.f. $\dot{k} = \|e\|^2$ in the adaptive context) but is simply a static, nonlinear (albeit time-varying), though memoryless feedback, and (iv) growth assumptions on the system nonlinearities are obviated.
Miller and Davison (Miller and Davison, 1991) have introduced a controller which guarantees the "error to be less than an (arbitrarily small) prespecified constant after an (arbitrarily small) prespecified period of time, with an (arbitrarily small) prespecified upper bound on the amount of overshoot." However, their controller is adaptive with monotonically non-decreasing gain, invokes a piecewise constant switching strategy, and is less flexible in its scope for shaping transient behaviour.

The paper is organised as follows. In Section 2, we make precise the underlying system class and provide some examples. The control problem is formulated in Section 3, wherein the class of reference signals and the performance funnel are described. Section 4 elucidates the proposed output feedback control and, in the main result (Theorem 2), establishes the requisite transient and asymptotic behaviour of the closed-loop system. Finally, in Section 5, the flexibility in the choice of gain functions $K_{\mathcal{F}}$, alluded to above, is illustrated via diverse examples determined by a variety of measures of distance to the funnel boundary.

We close this section with some remarks on notation. Throughout, $\mathbb{R}_{\geq 0} := [0, \infty)$, $\mathbb{R}_{>0} := (0, \infty)$, the inner product on $\mathbb{R}^M$ is $\langle x, y \rangle = x^T y$, the Euclidean norm on $\mathbb{R}^M$ is given by $\|x\| := \sqrt{x^T x}$, and $\mathbb{B}_\delta(\xi) := \left\{ x \in \mathbb{R}^n \middle| \, \|x - \xi\| < \delta \right\}$

is the open ball of radius $\delta > 0$ centred at $\xi \in \mathbb{R}^M$. The Euclidean distance of $x \in \mathbb{R}^M$ from a non-empty set $A \subset \mathbb{R}^M$ is $\operatorname{dist}(x, A) := \inf_{a\in A} \|x - a\|$. The space of continuous functions $S \to \mathbb{R}^M$ is denoted by $C(S;\mathbb{R}^M)$, $L^\infty(I;\mathbb{R}^M)$ is the space of measurable essentially bounded functions $I \to \mathbb{R}^M$ ($I \subset \mathbb{R}$ an interval), with norm, $\|x\|_\infty := \operatorname{ess\,sup}_{t\in I}\|x(t)\|$, $L^\infty_{\text{loc}}(I;\mathbb{R}^M)$ is the space of measurable, locally essentially bounded functions $I \to \mathbb{R}^M$, and finally $W^{1,\infty}(\mathbb{R}_{\geq 0};\mathbb{R}^M)$ denotes the space of bounded locally absolutely continuous functions $r : \mathbb{R}_{\geq 0} \to \mathbb{R}^M$ with essentially bounded derivative and norm $\|x\|_{1,\infty} := \|x\|_\infty + \|\dot{x}\|_\infty$.

2 System class Σ

Consider the class Σ of infinite-dimensional, nonlinear, M-input u, M-output y systems (p, f, T), given by a controlled nonlinear functional differential equation of the form

$$\dot{y}(t) = f\Big(p(t), (Ty)(t), u(t)\Big), \quad y_{[-h,0]} = y^0, \quad h \geq 0, \quad y^0 \in C\Big([-h, 0];\mathbb{R}^M\Big) \tag{6}$$

having the following properties for some $P, Q \in \mathbb{N}$:

1. $p \in L^\infty(\mathbb{R}_{\geq 0};\mathbb{R}^P)$;
2. $f \in C\Big(\mathbb{R}^P \times \mathbb{R}^Q \times \mathbb{R}^M;\, \mathbb{R}^M\Big)$;
3. for every non-empty compact subset $\mathcal{C} \subseteq \mathbb{R}^P \times \mathbb{R}^Q$ and every sequence (u_n) in $\mathbb{R}^M\backslash\{0\}$ the following property (akin to radial unboundedness or weak coercivity) holds:
$$\|u_n\| \to \infty \text{ as } n \to \infty \quad \Longrightarrow \quad \min_{(v,w)\in\mathcal{C}} \frac{\langle u_n, f(v, w, u_n)\rangle}{\|u_n\|} \to \infty \text{ as } n \to \infty\,;$$
4. $T : C([-h,\infty);\mathbb{R}^M) \to L^\infty_{\text{loc}}(\mathbb{R}_{\geq 0};\mathbb{R}^Q)$ denotes an operator of class $\mathcal{T}$, that is, an operator with the following three properties:
   (a) for all $\delta > 0$ there exists $\Delta > 0$ such that, for all $x \in C\Big([-h,\infty);\mathbb{R}^M\Big)$,
   $$\|x\|_\infty \leq \delta \implies \Big\|Tx\|_\infty \leq \Delta\,;$$
   (b) for all $t \geq 0$ and all $x, \xi \in C\Big([-h,\infty);\mathbb{R}^M\Big)$
   $$x|_{[-h,t]} = \xi|_{[-h,t]} \implies Tx|_{[0,t]} = T\xi|_{[0,t]}\,;$$
   (c) for all $t \geq 0$ and all $\zeta \in C\Big([-h, t];\mathbb{R}^M\Big)$ there exist $\tau, \delta, c > 0$ such that, for all $x, \xi \in C\Big([-h,\infty);\mathbb{R}^M\Big)$ with $x|_{[-h,t]} = \zeta = \xi|_{[-h,t]}$ and $x(s), \xi(s) \in \mathbb{B}_\delta(\zeta(t))$ for all $s \in [t, t + \tau]$,
   $$\|(Tx)(s) - (T\xi)(s)\| \;\leq\; c \sup_{s\in[t,t+\tau]} \|x(s) - \xi(s)\|\,.$$

Remark 1

(i) The function p in (6) may be thought of as a (bounded) disturbance term; the non-negative constant h quantifies the "memory" of the system.

(ii) Property 3 generalizes the positive "high-frequency gain" concept in linear systems of relative degree one.

(iii) It is straightforward to show that a necessary and sufficient condition for Property 3 to hold is that, for $\mathbb{S}^{M-1} := \{u \in \mathbb{R}^M | \ \|u\| = 1\}$ and for every compact set $\mathcal{C} \subset \mathbb{R}^P \times \mathbb{R}^Q$, the continuous function $\gamma_{\mathcal{C}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, defined below, has the following property:

$$\min_{(v,w,u)\in\mathcal{C}\times\mathbb{S}^{M-1}} \langle u, f(v, w, su)\rangle =: \gamma_{\mathcal{C}}(s) \to \infty \qquad \text{as} \qquad s \to \infty. \tag{7}$$

(iv) Property 4(a) is a crucial "bounded-input, bounded-output" assumption on the operator T (this generalizes the rôle of the minimum-phase condition in the context of linear systems).

(v) Property 4(b) is an assumption of causality and Property 4(c) is a technical assumption on T of a "locally Lipschitz" nature.

(vi) Let $T \in \mathcal{T}$ and $t \geq 0$. Given $x \in C([-h, t); \mathbb{R}^M)$ let x^e denote an arbitrary extension of x to $C([-h, \infty); \mathbb{R}^M)$. By virtue of Property 4(b), $Tx^e|_{[0,t)}$ is uniquely determined by the function x in the sense that, the former is independent of the extension x^e chosen for the latter. Expanding on this observation, we will adopt the following notational convention. For $s \in [0, t)$, we simply write $(Tx)(s)$ in place of $(Tx^e)(s)$ (where $x^e \in C([-h, \infty); \mathbb{R}^M)$ is any continuous extension of x).

In the remainder of this section, we present some examples of systems belonging to the class Σ.

The linear prototype. With reference to finite-dimensional, linear, minimum-phase systems of the form (1)–(3), positivity of $B + B^T$ ensures Property 3, and the assumption that A_4 is Hurwitz (minimum phase) ensures Property 4.

Infinite-dimensional linear systems. The class of finite-dimensional systems considered in (1) can be extended to an infinite-dimensional setting by reinterpreting the operators $A_1, \ldots, A_4$ in the system representation (1) as the generating operators of a regular linear system (regular in the sense of (Weiss, 1994)). In particular, in this setting, A_4 is assumed to be the generator of a strongly continuous semigroup $\mathbf{S} = (\mathbf{S}_t)_{t\geq 0}$ of bounded linear operators on a Hilbert space X with norm $\| \cdot \|_X$. Let X_1 denote the space dom(A_4) endowed with the graph norm and X_{-1} denotes the completion of X with respect to the norm $\|z\|_{-1} = \left\|(s_0 I - A_4)^{-1} z\right\|_X$ where s_0 is any fixed element of the resolvent set of A_4. Then A_3 is assumed to be a bounded linear operator from $\mathbb{R}^m$ to X_{-1} and A_2 is assumed to be a bounded linear operator from X_1 to $\mathbb{R}^m$. $A_1, B \in \mathbb{R}^{m\times m}$.

If we assume that the semigroup $\mathbf{S}$ is exponentially stable and that the operator A_2 extends to a bounded linear operator (again denoted by A_2) from X to $\mathbb{R}^m$, then the operator $(Ty)(t) := A_1 y(t) + A_2 \int_0^t \mathbf{S}_{t-s} A_3 y(s)\, ds$ has Property 4 (for details, see (Ryan and Sangwin, 2001)).

Nonlinear delay elements. Let functions $\Psi_n : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^q : (t, y) \mapsto \Psi_n(t, y)$, $n = 0, ..., N$, be measurable in t and globally Lipschitz in y uniformly with respect to t: precisely, (i) for each fixed y, $\Psi_n(\cdot, y)$ is measurable and (ii) there exists a constant c such that, for almost all t and all $y, z \in \mathbb{R}^m$, $\Big\|\Psi_n(t, y) - \Psi_n(t, z)\Big\| \leq c\, \|y - z\|$. Assume further that $\Psi_n(\cdot, 0) = 0$. For $n = 0, ..., N$, let $h_n \geq 0$ and define $h := \max_n h_n$. For $y \in \mathcal{C}([-h, \infty); \mathbb{R}^m)$, the operator T, defined, for all $t \geq 0$, by $(Ty)(t) := \int_{-h_0}^0 \Psi_0(s, y(t+s))\, ds + \sum_{n=1}^N \Psi_n(t, y(t-h_n))$, has Property 4 (for details, see (Ryan and Sangwin, 2001)).

Systems with hysteresis. A general class of nonlinear operators $C(\mathbb{R}_{\geq 0}; \mathbb{R}) \to C(\mathbb{R}_{\geq 0}; \mathbb{R})$, which includes many physically motivated hysteretic effects, is defined via assumptions (N1)–(N8) of (Logemann and Mawby, 2000, Sec. 3). These assumptions are covered by Assumption 4 of Sub-section 2. Examples of such operators, including relay hysteresis, backlash hysteresis, elastic-plastic hysteresis and Preisach operators, are detailed in (Logemann and Mawby, 2000, Sec. 5).

ISS systems. Further examples of interconnected nonlinear systems with operators T of the allowable class $\mathcal{T}$ generated by input-to-state stable subsystem dynamics can be found in (Ryan and Sangwin, 2001, Sec. 2.3).

3 Problem formulation

3.1 The performance funnel

Let Φ denote the class of functions $\varphi \in W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R})$ which are positive-valued on $(0, \infty)$ and bounded away from zero "at infinity", that is,

$$\Phi := \Big\{\varphi \in W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}) \,|\, \varphi(s) > 0 \text{ for all } s > 0 \;\text{ and }\; \liminf_{s \to \infty} \varphi(s) > 0\Big\}.$$

With $\varphi \in \Phi$, we associate a set-valued map (defined on $\mathbb{R}_{\geq 0}$)

$$t \mapsto F(t) := \Big\{e \in \mathbb{R}^M \,|\; \varphi(t)\|e\| < 1\Big\},$$

the graph of which we refer to as the performance funnel

$$\mathcal{F} := \text{graph}(F) := \Big\{(t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^M \,|\; e \in F(t)\Big\}.$$

Observe that (i) $\varphi(0) = 0$ is permissible, in which case, $F(0) = \mathbb{R}^M$, and (ii) for every $\varphi \in \Phi$ and $\tau > 0$, there exists $\mu > 0$ such that $\varphi(t) \geq \mu$ for all $t \geq \tau$,

and so $F(t) \subset \mathbb{B}_{1/\mu}(0)$ for all $t \geq \tau$.
As a concrete example, for $\lambda > 0$, $\tau > 0$ and $\varepsilon \in (0,1)$, the choice

$$t \mapsto \varphi(t) = \frac{t}{([1-\varepsilon]t + \varepsilon\tau)\lambda}$$

yields an associated performance funnel $\mathcal{F}$ which reflects an overall objective of attaining tracking accuracy λ in prescribed time τ.

3.2 Class of reference signals and control objective

As reference signals r, we allow bounded locally absolutely continuous functions with bounded derivative, i.e. $r \in W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^M)$ with norm given by $\|r\|_{1,\infty} := \|r\|_\infty + \|\dot{r}\|_\infty$.

Given $\varphi \in \Phi$ and its associated performance funnel $\mathcal{F}$, the control objective is a single feedback strategy ensuring that, for each reference signal $r \in W^{1,\infty}$ and every system of class Σ, the tracking error $e = y - r$ has graph in $\mathcal{F}$ (equivalently: $e(t) \in F(t)$ for all $t \geq 0$), and all variables are bounded.

4 Output feedback control

Let $\varphi \in \Phi$ determine a performance funnel $\mathcal{F}$ and let $r \in W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^M)$. We seek to achieve the above control objective via the simple proportional time-varying output error feedback

$$u(t) = -k(t)e(t), \quad k(t) = K_{\mathcal{F}}(t, e(t)), \quad e(t) = y(t) - r(t), \tag{8}$$

whilst ensuring boundedness of the gain k. Here, $K_{\mathcal{F}} : \mathcal{F} \to \mathbb{R}_{\geq 0}$ is a continuous function chosen to confirm the intuition underlying the control structure: $K_{\mathcal{F}}$ is such that, if $(t, e(t))$ approaches the boundary of the funnel $\mathcal{F}$, then the gain $k(t) = K_{\mathcal{F}}(t, e(t))$ increases at a rate sufficient to preclude – via an implicit high-gain stability property of underlying system class Σ – boundary contact, thereby maintaining the error evolution within the performance funnel. Next, we elucidate two properties which, when imposed on the gain function $K_{\mathcal{F}}$, confirm this intuition.

4.1 Requisite properties of the gain function

Let $\varphi \in \Phi$, with associated map $t \mapsto F(t)$ and performance funnel $\mathcal{F} = \text{graph}(F)$. For each $t \in \mathbb{R}_{\geq 0}$, we denote the boundary of the set $F(t)$ by

$\partial F(t)$. Let $K_{\mathcal{F}} : \mathcal{F} \to \mathbb{R}_{\geq 0}$ be a continuous function. We impose only the following additional properties on $K_{\mathcal{F}}$.

Property A: $\forall\, K > 0 \;\; \exists\, \varepsilon > 0 : \;\; \forall\, (t, e) \in \mathcal{F}$

$$\Big[\operatorname{dist}(e, \partial F(t)) \leq \varepsilon \quad \Longrightarrow \quad K_{\mathcal{F}}(t, e) \geq K \Big].$$

Property B: $\forall\, \varepsilon > 0 \;\forall\, \delta > 0 \;\exists\, K > 0 : \;\; \forall\, (t, e) \in \mathcal{F}$

$$\Big[\operatorname{dist}(e, \partial F(t)) \geq \varepsilon \quad \text{and} \quad t \geq \delta \quad \Longrightarrow \quad K_{\mathcal{F}}(t, e) \leq K \Big].$$

The essence of these properties is as follows. Property A ensures that, in (8), if the tracking error $e(t)$ is close to the funnel boundary, then the associated gain value $k(t)$ is large. Property B, loosely speaking, obviates the need for large gain values away from the funnel boundary.

4.2 The main result

We now arrive at the main result, the essence of which is the assertion that the control objective is achieved by the feedback (8) provided that $K_{\mathcal{F}}$ has Properties A and B; moreover, the function $k(\cdot)$ is bounded.

Theorem 2 Let $(f, p, T) \in \Sigma$. Let $\varphi \in \Phi$ with associated map F and performance funnel $\mathcal{F} = \operatorname{graph}(F)$. Let $K_{\mathcal{F}} : \mathcal{F} \to \mathbb{R}_{\geq 0}$ be continuous with Properties A and B.
For any $r \in W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^M)$ and initial data $y^0 \in C\big([-h, 0]; \mathbb{R}^M\big)$ such that $y^0(0) - r(0) \in F(0)$, there exists a solution of the closed-loop initial-value problem

$$\left.\begin{aligned} &\dot{y}(t) = f\Big(p(t), (Ty)(t), -K_{\mathcal{F}}(t, y(t) - r(t))[y(t) - r(t)]\Big), \\ &y(t) - r(t) \in F(t), \quad y|_{[-h,0]} = y^0 \,. \end{aligned}\right\} \tag{9}$$

Every solution can be extended to a maximal solution $y : [-h, \omega) \to \mathbb{R}^n$ and every maximal solution has the following properties

(i) $\omega = \infty$,
(ii) $t \mapsto k(t) = K_{\mathcal{F}}(t, y(t) - r(t))$ is bounded on $\mathbb{R}_{\geq 0}$,
(iii) there exists $\varepsilon > 0$ such that $\operatorname{dist}\big(y(t) - r(t), \partial F(t)\big) \geq \varepsilon$ for all $t \in \mathbb{R}_{\geq 0}$.

Proof: Let $(p, f, T) \in \Sigma$, $r \in W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^M)$ and $y^0 \in C([-h, 0]; \mathbb{R}^M)$ with $y^0(0) - r(0) \in F(0)$. By a solution of the feedback-controlled initial-value problem (9) we mean a function $y \in C([-h, \omega); \mathbb{R}^M)$, with $0 < \omega \leq \infty$ and $y_{[-h,0]} = y^0$, such that $y|_{[0,\omega)}$ is absolutely continuous and satisfies the differential equation in (9) for almost all $t \in [0, \omega)$ and $y(t) - r(t) \in F(t)$ for all $t \in [0, \omega)$; y is maximal if it has no proper right extension that is also a solution.

Step 1: We show existence of a solution of (9) and establish that every solution can be extended to a maximal solution.
Writing $e(t) := y(t) - r(t)$, introducing the artifact $z(t) = t$, extending r to $[-h, \infty)$ by defining $r(t) := r(0)$ for all $t \in [-h, 0]$, and writing $x^0 := \big(0\,,\, y^0 - r|_{[-h,0]}\big)$, system (9) may be expressed in the equivalent form

$$\left.\begin{aligned}
&\dot z(t) = 1,\\
&\dot e(t) = f\Big(p(t), (T(e+r))(t), -K_{K_{\mathcal F}}(z(t), e(t))\, e(t)\Big) - \dot r(t),\\
&(z(t), e(t)) \in \widehat{\mathcal F} := \Big\{(z,e) \in \mathbb R \times \mathbb R^M \,\Big|\, e \in F(|z|)\Big\},\\
&(z,e)|_{[-h,0]} = x^0 \in C([-h,0]; \mathbb R \times \mathbb R^M), \quad x^0(0) \in \widehat{\mathcal F}\,,
\end{aligned}\right\} \tag{10}$$

which, on writing $x(t) = (z(t), e(t))$, $(\widehat T x)(t) = (\widehat T(z,e))(t) := (T(e+r))(t)$, and

$$\begin{aligned}
&G : \mathbb R_{\ge 0} \times \widehat{\mathcal F} \times \mathbb R^Q \to \mathbb R^{M+1},\\
&\qquad (t, x, w) \mapsto G(t, (z,e), w) := \Big(1\,,\, f\Big(p(t), w, -K_{K_{\mathcal F}}(|z|, e)\, e\Big) - \dot r(t)\Big)\,,
\end{aligned}$$

can be interpreted as the initial-value problem

$$\left.\begin{aligned}
&\dot x(t) = G(t, x(t), (\widehat T x)(t)), \quad x(t) \in \widehat{\mathcal F},\\
&x|_{[-h,0]} = x^0 \in C([-h,0]; \mathbb R^{M+1})\,, \quad x^0(0) \in \widehat{\mathcal F}\,.
\end{aligned}\right\} \tag{11}$$

Now $\widehat{\mathcal F} \subset \mathbb R^{M+1}$ is a non-empty open set, $\widehat T$ is a causal operator of class $\mathcal T$ (for M replaced by $M+1$) and G is locally essentially bounded and is a Carathéodory function [1], and so we may apply (Ilchmann *et al.*, 2002*b*, Theorem 5) to conclude that (11) has a solution and every solution may be extended to a maximal solution $x = (z, e) : [-h, \omega) \to \widehat{\mathcal F}$. Furthermore, if $\omega < \infty$, then, for every compact $\mathcal C \subset \widehat{\mathcal F}$, there exists $t' \in [0, \omega)$ such that $x(t') \notin \mathcal C$. Since (9) and (10) are equivalent representations of the same initial-value problem, it follows that (9) has a solution and every solution can be maximally extended. If $y : [-h, \omega) \to \mathbb R^M$ is a maximal solution of (9), then

[1] That is: (i) $G(t, \cdot, \cdot)$ is continuous for each fixed $t \in \mathbb R$, (ii) $G(\cdot, x, w)$ is measurable for each fixed $(x, w) \in \widehat{\mathcal F} \times \mathbb R^Q$, and (iii) for each compact $\mathcal C \subset \widehat{\mathcal F} \times \mathbb R^Q$ there exists $\kappa \in L^1_{\mathrm{loc}}([-h, \infty); \mathbb R_{\ge 0})$ such that $\|G(t, x, w)\| \le \kappa(t)$ for almost all $t \in [-h, \infty)$ and all $(x, w) \in \mathcal C$.

graph$(y-r) \subset \mathcal{F} = $ graph(F); moreover,

$$\omega < \infty \quad \Longrightarrow$$
$$\forall \, \text{compact } \mathcal{C} \subset \mathcal{F} \;\; \exists \, t' \in [0,\omega): \quad \big(t', y(t') - r(t')\big) = \big(t', e(t')\big) \notin \mathcal{C}. \quad (12)$$

Let $y : [-h,\omega) \to \mathbb{R}^M$, $0 < \omega \leq \infty$ be a maximal solution of (9) and write $e = y - r$ (with graph$(e) \subset \mathcal{F}$).

Step 2: We highlight an essential inequality.
Let $\tau \in (0,\omega)$. By properties of F, there exists $\mu > 0$ such that $F(t) \subset \mathbb{B}_{1/\mu}(0)$ for all $t \geq \tau$. Since $e(t) \in F(t)$ for all $t \in [0,\omega)$, it follows that e is bounded which, in conjunction with boundedness of the reference signal r, implies boundedness of y. Since p is essentially bounded and $T \in \mathcal{T}$ satisfies Property 4a of the system class Σ, there exists a non-empty compact set $\mathcal{C} \subset \mathbb{R}^P \times \mathbb{R}^Q$ such that $\big(p(t), (Ty)(t)\big) \in \mathcal{C}$ for almost all $t \in [0,\omega)$. Let $\gamma_{\mathcal{C}}$ defined as in (7) (and so $\gamma_{\mathcal{C}}(s) \to \infty$ as $s \to \infty$). Then, by Property 3 of the system class Σ and essential boundedness of $\dot{r}$, there exists a constant $c_1 \geq 0$ (see (Ilchmann *et al.*, 2002*b*, (30),(31))) such that

$$\begin{aligned}\frac{d}{dt}\|e(t)\|^2 &= 2\Big\langle e(t), f\Big(p(t), (Ty)(t), -K_{\mathcal{F}}(t,e(t))e(t)\Big)\Big\rangle - \dot{r}(t) \\ &\leq -2\gamma_{\mathcal{C}}\Big(\|e(t)\| K_F(t,e(t))\Big) + c_1 \quad \text{for almost all } t \in [0,\omega)\,.\end{aligned}$$

By boundedness of φ and e, together with essential boundedness of $\dot{\varphi}$, we now infer the existence of a constant $c_2 > 0$ such that

$$\begin{aligned}\frac{d}{dt}\Big(\varphi(t)\|e(t)\|\Big)^2 &= \Big(\varphi(t)\Big)^2 \frac{d}{dt}\|e(t)\|^2 + 2\varphi(t)\dot{\varphi}(t)\|e(t)\|^2 \\ &\leq -2\varphi(t)^2\|e(t)\|\gamma_{\mathcal{C}}\Big(\|e(t)\| K_{\mathcal{F}}(t,e(t))\Big) + c_2 \quad \text{a.a. } t \in [0,\omega)\,. \quad (13)\end{aligned}$$

Step 3: We show that the function $\tilde{k} : [0,\omega) \to \mathbb{R}_{\geq 0}$, $t \mapsto (1 - \varphi(t)\|e(t)\|)^{-1}$, is bounded. Choose $\delta \in (0,\omega)$ arbitrarily. By continuity, $\tilde{k}$ is bounded on $[0,\delta]$. Seeking a contradiction, suppose $\tilde{k}$ is unbounded on $[\delta,\omega)$. For each $n \in \mathbb{N}$, define $\sigma_n := \sup\Big\{t \in [\delta,\omega) \mid \tilde{k}(t) = \tilde{k}(\delta) + n\Big\}$ and $\tau_n := \inf\Big\{t \in [\delta,\omega) \mid \tilde{k}(t) = \tilde{k}(\delta) + n + 1\Big\}$. Then

$$\tilde{k}(t) \geq n + \tilde{k}(\delta) \qquad \forall \, t \in [\sigma_n, \tau_n], \; \forall \, n \in \mathbb{N}.$$

Define $\underline{\varphi} := \inf_{t \geq \delta} \varphi(t)$. By properties of $\varphi \in \Phi$, it follows that $\underline{\varphi} > 0$ and so we may define a decreasing sequence (ε_n) in $\mathbb{R}_{\geq 0}$, with $\varepsilon_n \searrow 0$ as $n \to \infty$, by

$$\varepsilon_n := \frac{1}{\underline{\varphi}\,[n + \tilde{k}(\delta)]} \quad \forall \, n \in \mathbb{N}.$$

We now have

$$\operatorname{dist}\Big(e(t),\partial F(t)\Big) = \frac{1}{\varphi(t)} - \|e(t)\| \;=\; \frac{1}{\varphi(t)\,\tilde{k}(t)} \;\le\; \frac{1}{\underline{\varphi}\,[n+\tilde{k}(\delta)]} \tag{14}$$

$$\le \varepsilon_n \qquad \forall\, t \in [\sigma_n, \tau_n],\ \forall\, n \in \mathbb{N}\,. \tag{15}$$

Next, we claim that the sequence (K_n) in $\mathbb{R}_{\ge 0}$, given by

$$K_n := \min_{t\in[\sigma_n,\tau_n]} K_{\mathcal{F}}\Big(t, e(t)\Big) \quad \forall\, n \in \mathbb{N},$$

is unbounded. By Property A of the gain function $K_{\mathcal{F}}$, there exists a sequence $(\tilde{\varepsilon}_k)$ in $(0,\infty)$ such that

$$\forall (t,e) \in \mathcal{F}\ \forall k \in \mathbb{N} \quad \Big[\operatorname{dist}(e,\partial F(t)) \le \tilde{\varepsilon}_k \quad \Longrightarrow \quad K_{\mathcal{F}}(t,e) \ge k\ \Big]. \tag{16}$$

Since $\lim_{n\to\infty} \varepsilon_n = 0$, we may choose, for every $k \in \mathbb{N}$, some $n_k \in \mathbb{N}$ such that $\varepsilon_{n_k} \le \tilde{\varepsilon}_k$. In view of (14) and (16), it follows that

$$K_{\mathcal{F}}\Big(t, e(t)\Big) \ge K_{n_k} \ge k \quad \forall\, t \in [\sigma_{n_k}, \tau_{n_k}],\ \forall\, k \in \mathbb{N}, \tag{17}$$

and so the sequence (K_n) has an unbounded subsequence, whence the claim.

By boundedness of φ, convergence to zero of the decreasing sequence (ε_n), and (14), we conclude the existence of constants $c_3 > 0$ and $\hat{n} \in \mathbb{N}$ such that

$$\|e(t)\| \ge \frac{1}{\varphi(t)} - \varepsilon_n \ge c_3 \quad \forall\, t \in [\sigma_n, \tau_n]\ \ \forall\, n \ge \tilde{n}. \tag{18}$$

Now by (13), together with (18), (17), unboundedness of (K_n) and the fact that $\gamma_{\mathcal{C}}(s) \to \infty$ as $s \to \infty$ (recall (7)), we may choose some $\hat{n} \ge \tilde{n}$ such that

$$\frac{d}{dt}\Big(\varphi(t)\|e(t)\|\Big)^2 < -2\underline{\varphi}^2 c_3 \gamma_{\mathcal{C}}\Big(\|e(t)\| K_{\mathcal{F}}(t,e(t)\Big) + c_2 < 0 \ \text{ for a.a. } t \in [\sigma_{\hat{n}}, \tau_{\hat{n}}],$$

whence the contradiction: $1+\tilde{k}(\sigma_{\hat{n}}) = \tilde{k}(\tau_{\hat{n}}) = \varphi(\tau_{\hat{n}})\|e(\tau_{\hat{n}})\| < \varphi(\sigma_{\hat{n}})\,\|e(\sigma_{\hat{n}})\| = \tilde{k}(\sigma_{\hat{n}})$. Therefore, $\tilde{k}$ is unbounded.

Step 4: We show $t \mapsto K_{\mathcal{F}}(t, e(t))$ is bounded on $[0,\omega)$.
Let $\delta \in (0,\omega)$. By continuity, $K_{\mathcal{F}}(\cdot, e(\cdot))$ is bounded on $[0,\delta]$. For contradiction, suppose that $K_{\mathcal{F}}$ is unbounded on $[\delta,\omega)$. Then there exists a sequence (t_n) in $[\delta,\omega)$ such that $K_{\mathcal{F}}\Big(t_n, e(t_n)\Big) \to \infty$ as $n \to \infty$.
We claim that

$$\liminf_{n\to\infty} \varepsilon_n = 0, \qquad \text{where}\ \ \varepsilon_n := \operatorname{dist}\Big(e(t_n), \partial F(t_n)\Big) > 0. \tag{19}$$

Suppose otherwise, then there exists $\varepsilon > 0$ such that $\varepsilon_n > \varepsilon$ for all $n \in \mathbb{N}$. By Property B of the gain function, there exists $K \ge 0$ such that

$$K_{\mathcal{F}}(t_n, e(t_n)) \le K \quad \text{ for all } n \in \mathbb{N}\,,$$

contradicting unboundedness of the sequence $\Big(K_{\mathcal{F}}(t_n, e(t_n))\Big)$. This establishes (19). Now, observe that, for all $n \in \mathbb{N}$,

$$\tilde{k}(t_n) = \frac{1}{1-\varphi(t_n)\|e(t_n)\|} = \frac{1}{\varphi(t_n)\operatorname{dist}(e(t_n), \partial F(t_n))} = \frac{1}{\varphi(t_n)\varepsilon_n} \geq \frac{1}{\|\varphi\|_\infty \varepsilon_n}.$$

which, in view of (19), contradicts boundeness of $\tilde{k}$. Therefore, the function $K_{\mathcal{F}}(\cdot, e(\cdot))$ is bounded on $[0, \omega)$.

Step 5: We show that there exists $\varepsilon > 0$ so that $\operatorname{dist}(e(t), \partial F(t)) \geq \varepsilon$ for all $t \in [0, \omega)$.
Suppose otherwise. Then there exists a sequence (t_n) in $[0, \omega)$ such that

$$\operatorname{dist}\Big(e(t_n), \partial F(t_n)\Big) \leq 1/n \quad \forall\, n \in \mathbb{N}\,.$$

By boundedness of $K_{\mathcal{F}}(\cdot, e(\cdot))$, $K := \sup_{t\in[0,\omega)} K_{\mathcal{F}}(t, e(t))$ is in $\mathbb{R}_{\geq 0}$. By Property A of the gain function $K_{\mathcal{F}}$, there exists $\hat{\varepsilon} > 0$ such that, for all $(t, e) \in \mathcal{F}$,

$$\operatorname{dist}(e, \partial F(t)) \leq \hat{\varepsilon} \quad \Longrightarrow \quad K_{\mathcal{F}}(t, e) > K\,.$$

Choosing $\hat{n} \in \mathbb{N}$ sufficiently large so that $\operatorname{dist}\Big(e(t_{\hat{n}}), \partial F(t_{\hat{n}})\Big) \leq 1/\hat{n} < \hat{\varepsilon}$ yields the contradiction

$$K_{\mathcal{F}}\Big(t_{\hat{n}}, e(t_{\hat{n}})\Big) \;> K = \sup_{t\in[0,\omega)} K_{\mathcal{F}}\Big(t, e(t)\Big)\,.$$

Step 6: Seeking a contradiction suppose $\omega < \infty$. Let $\delta \in (0, \omega)$ and $\varepsilon > 0$ be as in the claim of Step 5, in which case, $\varepsilon \leq 1/\varphi(t)$ for all $t \in [\delta, \omega]$. Define

$$\begin{aligned} \mathcal{C}_\delta &:= \Big\{(t, e) \in [\delta, \omega] \times \mathbb{R}^M \,\Big|\; e \in F(t),\; \operatorname{dist}(e, \partial F(t)) \geq \varepsilon\} \\ &= \Big\{(t, e) \in [\delta, \omega] \times \mathbb{R}^M \,\Big|\, \|e\| \leq \frac{1}{\varphi(t)} - \varepsilon\Big\}\,. \end{aligned}$$

Then $\mathcal{C}_\delta$ is compact. Now define the compact set $\tilde{\mathcal{C}} := \{(t, e(t)) |\; t \in [0, \delta]\}$. Then $\mathcal{C} = \tilde{\mathcal{C}} \cup \mathcal{C}_\delta$ is a compact subset of $\mathcal{F}$ with $(t, e(t)) \in \mathcal{C}$ for all $t \in [0, \omega)$ which contradicts property (12). Therefore, $\omega = \infty$.

Step 7: Finally, Step 6 together with Step 4 and 5 shows Assertions 1–3.
The proof of the theorem is therefore complete. □

5 Gain functions

In this section we describe various choices of continuous gain function $K_{\mathcal{F}}$, with the requisite Properties A and B, which are feasible for the feedback

(8) and which are based on different "measures" of distance to the funnel boundary.

5.1 *Scaled vertical distance to the funnel boundary*

Here, we base the gain function on measurements of the distance of the instantaneous error $e(t)$ from the boundary of the set $F(t)$: this approach uses only funnel information at current time t and, in particular, does not anticipate the future shape of the funnel boundary.

With reference to Figure 3, for $(t,e) \in \mathcal{F}$, we refer to $\operatorname{dist}(e, \partial F(t)) = 1/\varphi(t) - \|e\|$ (with the convention that $\operatorname{dist}(e, \partial F(0)) = \infty$ if $\varphi(0) = 0$) as the vertical distance from (t,e) to the funnel boundary: in incorporating this distance in the design of gain functions $K_{\mathcal{F}}$, we allow for scaling by a suitable function ψ and refer to the quantity $\psi(t)\operatorname{dist}(e, \partial F(t))$ as a scaled vertical distance.

Proposition 3 Let $\varphi, \psi \in \Phi$ such that $\lim_{t\to 0+} \psi(t)\,\varphi(t)^{-1} =: \psi_0 \in (0,\infty]$, and let $\mathcal{F}$ be the performance funnel associated with φ. Assume that $\beta : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ is continuous, unbounded and non-increasing. Then

$$K_{\mathcal{F}} : \mathcal{F} \to \mathbb{R}_{\geq 0},\ (t,e) \mapsto \begin{cases} \beta\Big(\psi(t)\operatorname{dist}(e, \partial F(t))\Big), & t > 0 \\ \beta\Big(\psi_0 - \psi(0)\|e\|\Big), & t = 0 \text{ and } \psi_0 < \infty \\ \beta_* := \lim_{s\to\infty} \beta(s), & t = 0 \text{ and } \psi_0 = \infty \end{cases} \tag{20}$$

is continuous and has Properties A and B (as in Subsection 4.1). □

Remark 4

(i) The simplest example, covered by Proposition 3, is the unscaled vertical distance: for $\psi \equiv 1$ and $\beta : s \mapsto 1/s$, we have, for all $(t,e) \in \mathcal{F}$,

$$K_{\mathcal{F}}(t,e) = \frac{1}{\operatorname{dist}(e, \partial F(t))} = \frac{\varphi(t)}{1 - \varphi(t)\|e\|} \tag{21}$$

(ii) The strategy introduced in (Ilchmann *et al.*, 2002*b*) is also covered by a function $K_{\mathcal{F}}$ satisfying Properties A and B. In (Ilchmann *et al.*, 2002*b*), the control gain is defined, for any $\varphi \in \Phi$ and corresponding funnel $\mathcal{F}$, as

$$k(t) = \alpha(\varphi(t)\|e(t)\|),$$

where $\alpha : [0,1) \to \mathbb{R}_{\geq 0}$ is some continuous, unbounded injection. Adopting the scaling $\psi = \varphi$ and introducing the continuous, unbounded and strictly decreasing function

$$\beta : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0},\ s \mapsto \beta(s) = \begin{cases} \alpha(1-s), & s \in (0,1] \\ \alpha(0), & s \geq 1, \end{cases}$$

we may interpret the above strategy in terms of a gain function of form (20) as follows

$$k(t) = K_{\mathcal{F}}(t, e(t)), \quad K_{\mathcal{F}}(t,e) := \begin{cases} \beta\Big(\varphi(t)\mathrm{dist}(e, \partial F(t))\Big), & (t,e) \in \mathcal{F},\ t > 0 \\ \beta\Big(1 - \varphi(0)\|e\|\Big), & (t,e) \in \mathcal{F},\ t = 0. \end{cases}$$

In this case, the scaling of the vertical distance by the special choice $\psi = \varphi$ is restrictive: Proposition 3 offers considerably more flexibility in the choice of scaling functions.

(iii) For technical reasons it is convenient to associate with β the "generalized inverse"

$$\beta^\dagger : (\beta_*, \infty) \to \mathbb{R}_{>0}, \quad s \mapsto \min\{\sigma \in \mathbb{R}_{>0} |\ \ \beta(\sigma) = s\}$$

with the properties

$$\beta(\beta^\dagger(s)) = s \quad \forall\ s \in (\beta_*, \infty) \quad \text{and} \quad \lim_{s\to\infty} \beta^\dagger(s) = 0.$$

Proof of Proposition 3: First, we prove continuity of $K_{\mathcal{F}}$ on $\mathcal{F}$. Continuity of $K_{\mathcal{F}}$ at points $(t,e) \in \mathcal{F}$ with $t > 0$ is an immediate consequence of continuity of the functions ψ, φ and dist, together with the fact that $\varphi(t) \neq 0$. It remains to prove continuity of $K_{\mathcal{F}}$ at points $(0,e) \in \mathcal{F}$. Let $(0,e) \in \mathcal{F}$ and let $\big((t_n, e_n)\big)$ be a sequence in $\mathcal{F}$ with $(t_n, e_n) \to (0,e)$ as $n \to \infty$ with $(t_n, e_n) \neq (0,e)$ for all $n \in \mathbb{N}$. Define

$$N^0 := \{n \in \mathbb{N} \mid t_n = 0\}, \qquad N^+ := \{n \in \mathbb{N} \mid t_n > 0\}.$$

If N^0 is infinite, then

$$\lim_{n\to\infty,\, n\in N^0} K_{\mathcal{F}}(t_n, e_n) = \left\{ \begin{array}{ll} \lim_{n\to\infty} \beta\Big(\psi_0 - \psi(0)\|e_n\|\Big), & \psi_0 < \infty \\ \beta_*, & \psi_0 = \infty \end{array} \right\} = K_{\mathcal{F}}(0,e)\,.$$

If N^+ is infinite, then

$$\lim_{n\to\infty,\, n\in N^+} K_{\mathcal{F}}(t_n, e_n) = \lim_{n\to\infty,\, n\in N^+} \beta\left(\frac{\psi(t_n)}{\varphi(t_n)} - \psi(t_n)\|e_n\|\right) = K_{\mathcal{F}}(0,e)$$

It now follows that

$$\lim_{n\to\infty} K_{\mathcal{F}}(t_n, e_n) = K_{\mathcal{F}}(0,e),$$

and so $K_{\mathcal{F}}$ is continuous at all points $(0, e) \in \mathcal{F}$.

Next, we establish Property A. Let $K > 0$ arbitrary and define, for $\beta^\dagger$ as in Remark 4(iii),

$$\varepsilon := \beta^\dagger(K + \beta_*)/\|\psi\|_\infty > 0\,.$$

Observe that, if $\mathrm{dist}(e, \partial F(0)) \le \varepsilon$, then $\varphi(0) > 0$ and $\psi(0)\mathrm{dist}(e, \partial F(0)) = \psi_0 - \psi(0)\|e\|$. We may now conclude that, for each $(t, e) \in \mathcal{F}$,

$$\begin{aligned} \mathrm{dist}(e,\partial F(t)) \le \varepsilon \quad &\Longrightarrow \quad \psi(t)\mathrm{dist}(e, \partial F(t)) \;\le\; \varepsilon\|\psi\|_\infty \;=\; \beta^\dagger(K + \beta_*) \\ &\Longrightarrow \quad K_{\mathcal{F}}(t, e) = \beta\Big(\psi(t)\mathrm{dist}(e, \partial F(t))\Big) \ge \beta\Big(\beta^\dagger(K + \beta_*)\Big) \ge K\,, \end{aligned} \tag{22}$$

and so Property A holds.

Finally, we establish Property B. Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary and define

$$K := \beta\Big(\min\{b, \varepsilon\underline{\psi}\}\Big) \quad \text{with} \quad \underline{\psi} := \inf_{t \ge \delta} \psi(t).$$

Let $(t, e) \in \mathcal{F}$. Then,

$$\begin{aligned} \mathrm{dist}(e, \partial F(t)) \ge \varepsilon \;\;\&\;\; t \ge \delta \quad &\Longrightarrow \quad \psi(t)\mathrm{dist}(e, \partial F(t)) \;\ge\; \varepsilon\underline{\psi} \;\ge\; \min\{b, \varepsilon\underline{\psi}\} \\ &\Longrightarrow \quad K_{\mathcal{F}}(t, e) \;=\; \beta\Big(\psi(t)\mathrm{dist}(e, F(t))\Big) \;\le\; K. \end{aligned}$$

This completes the proof. □

5.2 Distance to the future funnel boundary

As already mentioned, the scaled vertical distance, investigated in the previous subsection, uses only instantaneous funnel information. It is of theoretical interest, and also of relevance in certain applications, to incorporate anticipation of the future funnel shape in determining the current gain value. To this end, we next investigate the adoption of the distance $d_f(t, e)$ of $(t, e) \in \mathcal{F}$ to the future funnel boundary in the design of gain functions $K_{\mathcal{F}}$ with Properties A and B. For $\varphi \in \Phi$, with associated map F and performance funnel $\mathcal{F}$, this distance is defined, with reference to Figure 3, as follows

$$d_f : \mathcal{F} \to \mathbb{R}_{>0}, \quad (t, e) \mapsto \inf_{\tau > t} \sqrt{(\tau - t)^2 + \Big(\mathrm{dist}(e, \partial F(\tau)\Big)^2}.$$

In contrast with the (scaled) vertical distance of the previous subsection (which is infinite at $(0, e)$ in cases where $\varphi(0) = 0$), the distance $d_f(t, e)$ is finite for all $(t, e) \in \mathcal{F}$.

Proposition 5 Let $\varphi \in \Phi$, with associated map F and performance funnel $\mathcal{F}$, and let $\psi \in \Phi$ be such that $\psi(0) > 0$. Assume that $\beta : \mathbb{R}_{>0} \to \mathbb{R}_{\ge 0}$ is continuous, unbounded and non-increasing. Then the mappings $d_f : \mathcal{F} \to \mathbb{R}_{>0}$ and

$$K_{\mathcal{F}} : \mathcal{F} \to \mathbb{R}_{\geq 0}, \quad (t,e) \mapsto \beta\Big(\psi(t) d_f(t,e)\Big)$$

are continuous and $K_{\mathcal{F}}$ has Properties A and B.

Proof: We first show continuity of d_f.
Define

$$M(s) := \{(r, 1/\varphi(r)) | \; r > s\} \quad \text{for } s \geq 0\,,$$

and note that $d_f(t,e) = \operatorname{dist}((t, \|e\|), M(t))$ for all $(t,e) \in \mathcal{F}$. We will prove continuity of d_f by showing that the map $(t,e) \mapsto \operatorname{dist}((t,\|e\|), M(t))$ is continuous on $\mathcal{F}$. Let $(t,e) \in \mathcal{F}$ be arbitrary. For notational convenience, we introduce $\eta := (t, \|e\|)$ and

$$\theta : \mathbb{R}_{\geq 0} \to [0,\infty), \quad \tau \mapsto \theta(\tau) := \sqrt{\Big(\varphi(\tau)(t-\tau)\Big)^2 + \Big(1 - \varphi(\tau)\|e\|\Big)^2}\,.$$

The following is readily seen:

$$\forall\, s \geq 0 \;\exists\; \tau \geq s : \quad \varphi(\tau) > 0 \;\text{ and }\; \operatorname{dist}(\eta, M(s)) \;=\; \frac{\theta(\tau)}{\varphi(\tau)}\,.$$

Now consider the case wherein $\varphi(0) > 0$. Let $s \geq 0$ and $\varepsilon > 0$ be arbitrary. By continuity of φ, there exists $\delta \in (0, \varepsilon/2)$ such that

$$\sigma_1, \sigma_2 \in (s-\delta, s+\delta) \cap [0,\infty) \quad \Longrightarrow \quad |1/\varphi(\sigma_1) - 1/\varphi(\sigma_2)| < \varepsilon/2.$$

Let $\sigma \geq 0$ be such that $|\sigma - s| < \delta$. Let $\rho_0 := \min\{\sigma, s\}$ and $\rho_1 := \max\{\sigma, s\}$. Let $\tau \geq \rho_0$ be such that $\operatorname{dist}(\eta, M(\rho_0)) = \theta(\tau)/\varphi(\tau)$. Since $M(\rho_1) \subset M(\rho_0)$, it follows that $\operatorname{dist}(\eta, M(\rho_0)) \leq \operatorname{dist}(\eta, M(\rho_1))$, with equality holding if $\tau \geq \rho_1$ (in which case, we have $|\operatorname{dist}(\eta, M(\sigma)) - \operatorname{dist}(\eta, M(s))| = |\operatorname{dist}(\eta, M(\rho_1) - \operatorname{dist}(\eta, M(\rho_0))| = 0$). Moreover, if $\tau < \rho_1$, then $|\rho_1 - \tau| < |\sigma - s| < \delta$ and

$$\begin{aligned}|\operatorname{dist}(\eta, M(\sigma)) - \operatorname{dist}(\eta, M(s))| &= |\operatorname{dist}(\eta, M(\rho_1) - \operatorname{dist}(\eta, M(\rho_0))| \\ &= \operatorname{dist}(\eta, M(\rho_1)) - \theta(\tau)/\varphi(\tau) \leq \theta(\rho_1)/\varphi(\rho_1) - \theta(\tau)/\varphi(\tau) \\ &\leq \sqrt{(\rho_1 - \tau)^2 + (1/\varphi(\rho_1) - 1/\varphi(\tau))^2} \leq \sqrt{\delta^2 + (\varepsilon/2)^2} < \varepsilon\,.\end{aligned}$$

This completes the proof of continuity (on $\mathbb{R}_{\geq 0}$) of the map $s \mapsto \operatorname{dist}(\eta, M(s))$ in the case of $\varphi(0) > 0$. Next, we consider the case wherein $\varphi(0) = 0$. In this case, the above argument applies *mutatis mutandis* to conclude continuity of the map $\operatorname{dist}(\eta, M(\cdot))$ on the open interval $(0,\infty)$. It remains only to prove continuity at $s = 0$. Let $s = 0$. Then there exists $\tau > 0$ such that

$$\operatorname{dist}(\eta, M(\sigma)) = \operatorname{dist}(\eta, M(0)) = \theta(\tau)/\varphi(\tau) \quad \forall\; \sigma \in [0,\tau]$$

whence continuity at $s = 0$.

We proceed to prove continuity of d_f at $(t,e) \in \mathcal{F}$. Let $\varepsilon > 0$ be arbitrary. By continuity of the map $s \mapsto \mathrm{dist}(\eta, M(s))$, there exists $\delta_1 > 0$ such that, for all $s \geq 0$,

$$|s-t| < \delta_1 \quad \Longrightarrow \quad |\mathrm{dist}(\eta, M(t)) - \mathrm{dist}(\eta, M(s))| < \varepsilon/2$$

Since, for each $s \geq 0$, the map $\eta \mapsto \mathrm{dist}(\eta, M(s))$ is globally Lipschitz, with Lipschitz constant 1, it follows that

$$|\mathrm{dist}(\mu, M(s)) - \mathrm{dist}(\eta, M(s))| \leq \|\mu - \eta\| \quad \forall\, \mu \in \mathbb{R}^2 \quad \forall\, s \geq 0.$$

Now define $\delta := \min\{\delta_1, \varepsilon/2\}$. Then, for all $(s,v) \in \mathcal{F}$ with $\|(s,v)-(t,e)\| < \delta$, we have

$$\begin{aligned} |\mathrm{dist}((s,\|v\|), M(s)) - &\mathrm{dist}((t,\|e\|), M(t))| \\ &\leq |\mathrm{dist}((t,\|e\|), M(s)) - \mathrm{dist}((t,\|e\|), M(t))| \\ &\qquad + |\mathrm{dist}((s,\|v\|), M(s)) - \mathrm{dist}((t,\|e\|), M(s))| \\ &\leq \varepsilon/2 + \delta \leq \varepsilon\,. \end{aligned}$$

This shows continuity of d_f.

Now continuity of $K_{\mathcal{F}}$ is a consequence of continuity of β and d_f.

Next, we prove Property A. Let $\beta^\dagger$ be as in Remark 4(iii). Let $K > 0$ be arbitrary and define $\varepsilon := \beta^\dagger(K + \beta_*)/\|\psi\|_\infty$. Let $(t,e) \in \mathcal{F}$. Then, we have

$$\begin{aligned} \mathrm{dist}(e, \partial F(t)) < \varepsilon \quad \Longrightarrow \quad d_f(t,e) < \varepsilon \quad \Longrightarrow \quad \psi(t) d_f(t,e) < \beta^\dagger(K+\beta_*) \\ \Longrightarrow K_{\mathcal{F}}(t,e) = \beta\big(\psi(t) d_f(t,e)\big) \geq \beta\big(\beta^\dagger(K+\beta_*)\big) \geq K\,, \end{aligned}$$

and so Property A holds.

It remains to prove Property B. Seeking a contradiction, suppose Property B fails to hold. Then there exist $\varepsilon > 0$, $\delta > 0$ and a sequence (t_n, e_n) in $\mathcal{F}$ such that $\mathrm{dist}(e_n, \partial F(t_n)) \geq \varepsilon$, $t_n \geq \delta$ and $K_{\mathcal{F}}(t_n, e_n) > n + \beta_*$ for all $n \in \mathbb{N}$.
For each $n \in \mathbb{N}$, define

$$\varepsilon_n := \beta^\dagger(n+\beta_*)/\underline{\psi} \qquad \text{with} \quad \underline{\psi} := \inf_{t \geq \delta} \psi(t) > 0\,.$$

It now follows that

$$\begin{aligned} K_{\mathcal{F}}(t_n, e_n) = \beta(\psi(t_n) d_f(t_n, e_n)) > n + \beta_* \quad \Longrightarrow \quad \psi(t_n) d_f(t_n, e_n) \leq \beta^\dagger(n+\beta_*) \\ \Longrightarrow \quad d_f(t_n, e_n) \leq \beta^\dagger(n+\beta_*)/\underline{\psi} = \varepsilon_n \quad \forall\, n \in \mathbb{N}\,. \end{aligned}$$

Therefore, for each $n \in \mathbb{N}$, there exists $(\tau_n, z_n) \in \mathbb{R}_{>0} \times \partial F(\tau_n)$, with $\tau_n \geq t_n$ and $\|z_n\| = 1/\varphi(\tau_n)$, such that $\|(t_n, e_n) - (\tau_n, z_n)\| < 2\varepsilon_n$. Now, since $\varphi \in W^{1,\infty}$, the reciprocal function $1/\varphi(\cdot)$ satisfies a global Lipschitz condition (with Lipschitz constant L) on $[\delta, \infty)$. We now arrive at a contradiction

$$\begin{aligned} 0 < \varepsilon \leq \mathrm{dist}(e_n, \partial F(t_n)) = \frac{1}{\varphi(t_n)} - \|e_n\| \leq \left| \frac{1}{\varphi(t_n)} - \frac{1}{\varphi(\tau_n)} \right| + \Big| \|z_n\| - \|e_n\| \Big| \\ \leq \; L|t_n - \tau_n| + \|z_n - e_n\| \; \leq \; 2[L+1]\varepsilon_n \to 0 \;\; \text{as } n \to \infty\,. \end{aligned}$$

Therefore, Property B holds. This completes the proof of the proposition. □

5.3 A numerical future distance to the funnel boundary

In applications, the distance function d_f of the previous sub-section may prove difficult to realize. The following distance function is simpler to compute and, loosely speaking, may be regarded as a numerical approximation to d_f. For

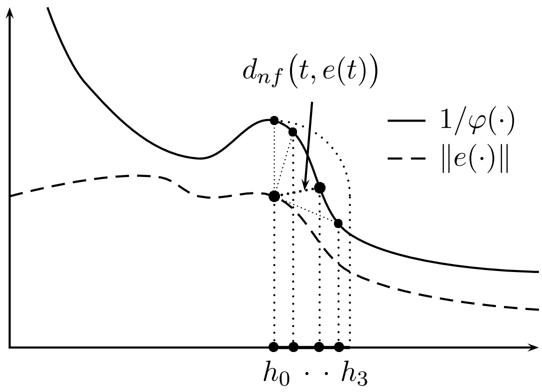

Fig. 4. The numerical distance d_{nf} to the future funnel boundary.

$N \in \mathbb{N}$, choose a partition of $[0, 1]$

$$0 = h_0 \ < \ h_1 \ < \ \dots \ < \ h_N \leq 1.$$

Let $\varphi \in \Phi$ such that $\varphi(0) > 0$, and let $\mathcal{F}$ be the associated performance funnel. For notational simplicity, we write

$$d(t, e) := \operatorname{dist}(e, \partial F(t)) < \infty \qquad \text{for all } (t, e) \in \mathcal{F}\,.$$

The numerical future distance to the funnel boundary is the function $d_{nf} : \mathcal{F} \to \mathbb{R}_{>0}$ given by

$$\begin{aligned} d_{nf}(t, e) &:= \min_{i \in \{0,\dots,N\}} \operatorname{dist}\Big((t, \|e\|), (t + h_i d(t, e), 1/\varphi(t + h_i d(t, e))\Big) \\ &= \min_{i \in \{0,\dots,N\}} \sqrt{\Big(h_i d(t, e)\Big)^2 + \left(\frac{1}{\varphi(t + h_i d(t, e))} - \|e\|\right)^2}\,. \end{aligned} \tag{23}$$

The numerical future distance calculates, at any time t, the distance to the funnel boundary at finitely many future points $t + h_i d(t, e)$. Observe that, since $\operatorname{dist}\Big((t, \|e\|), (t + \delta, 1/\varphi(t + \delta)\Big) \geq \delta$ for all $\delta > 0$, it is not necessary to look further into the future than the value of the actual "vertical" distance $\operatorname{dist}(e, \partial F(t)) = d(t, e)$: this observation justifies the adoption of of the interval $[0, 1]$ for partition.

Proposition 6 Let φ, $\psi \in \Phi$ with $\varphi(0) > 0$ and $\psi(0) > 0$. Let $\mathcal{F}$ be the performance funnel associated with φ and assume that $\beta : (0,\infty) \to \mathbb{R}_{\geq 0}$ is a continuous, non-increasing and unbounded function. Then

$$K_{\mathcal{F}} : \mathcal{F} \to \mathbb{R}_{\geq 0}, \quad (t,e) \mapsto \beta\Big(\psi(t) d_{nf}(t,e)\Big)$$

is continuous and satisfies the Properties A and B in Sub-section 4.1.

Proof: Since $(t,e) \mapsto d(t,e) = \operatorname{dist}(e, \partial F(t))$ is continuous on $\mathcal{F}$, the functions

$$(t,e) \mapsto \Big(h_i d(t,e)\Big)^2 + \left(\frac{1}{\varphi(t + h_i d(t,e))} - \|e\|\right)^2, \qquad i = 0,1,\ldots,N$$

are continuous on $\mathcal{F}$. Therefore d_{nf} is continuous as a minimum of finitely many continuous functions and continuity of $K_{\mathcal{F}}$ follows from continuity of d_{nf}, ψ, and β.

Next, we establish Property A. For $\beta^\dagger$ as in Remark 4(iii) and $K > 0$, we have

$$\begin{aligned}(t,e) \in \mathcal{F}, \quad K_{\mathcal{F}}(t,e) < K \quad &\Longrightarrow \\ \varepsilon := \frac{\beta^\dagger(K)}{\|\psi\|_\infty} < d_{nf}(t,e) &\leq \operatorname{dist}\Big((t,\|e\|),(t,1/\varphi(t))\Big) = \operatorname{dist}(e, \partial F(t)),\end{aligned}$$

whence Property A. Finally, we establish Property B. Seeking a contradiction, suppose there exist $\varepsilon > 0$, $\delta > 0$ and a sequence $(t_n, e_n) \in \mathcal{F}^{\mathbb{N}}$ such that

$$\operatorname{dist}(e_n, \partial F(t_n)) \geq \varepsilon, \quad t_n \geq \delta, \quad K_{\mathcal{F}}(t_n, e_n) > n \quad \forall\, n \in \mathbb{N}\,.$$

By definition of $K_{\mathcal{F}}$,

$$K_{\mathcal{F}}(t_n,e_n) > n \quad \Longrightarrow \quad d_{nf}(t_n,e_n) < \varepsilon_n := \frac{\beta^\dagger(n)}{\inf_{t\geq\delta}\psi(t)}, \quad \forall\, n \in \mathbb{N}\,.$$

For every $n \in \mathbb{N}$, choose $i_n \in \{0,1,\ldots,N\}$ such that

$$\Big(d_{nf}(t_n,e_n)\Big)^2 = \Big(h_{i_n} d(t_n,e_n)\Big)^2 + \left(\frac{1}{\varphi(t + h_{i_n} d(t_n,e_n))} - \|e_n\|\right)^2.$$

Note that

$$\begin{aligned}\varepsilon \leq \operatorname{dist}(e_n, \partial F(t_n)) &= d(t_n,e_n) \\ &= \sqrt{\Big(h_0 d(t_n,e_n)\Big)^2 + \left(\frac{1}{\varphi(t + h_0 d(t_n,e_n))} - \|e_n\|\right)^2}\,.\end{aligned}$$

Since $\lim_{n\to\infty} \varepsilon_n = 0$ and $d_{nf}(t_n, e_n) < \varepsilon_n$, there exists $\hat{n} \in \mathbb{N}$ such that $i_n \geq 1$ for all $n \geq \hat{n}$ and so

$$\varepsilon_n > d_{nf}(t_n, e_n) = \sqrt{\Big(h_{i_n} d(t_n, e_n)\Big)^2 + \left(\frac{1}{\varphi(t + h_{i_n} d(t_n, e_n))} - \|e_n\|\right)^2}$$
$$\geq h_{i_n} d(t_n, e_n) \geq h_1 \varepsilon \quad \forall\, n \geq \hat{n}\,.$$

This is a contradiction, and therefore the proof of the proposition is complete. □

References

Ilchmann, A., E.P. Ryan and C.J. Sangwin (2002*a*). Systems of controlled functional differential equations and adaptive tracking. *SIAM J. of Control and Optim.* **40**, 1746–1764.

Ilchmann, A., E.P. Ryan and C.J. Sangwin (2002*b*). Tracking with prescribed transient behaviour. *ESAIM Control, Opt. and Calculus of Variations* **7**, 471–493.

Logemann, H. and A.D. Mawby (2000). Low-gain integral control of infinite dimensional regular linear systems subject to input hysteresis. In: *Advances in Mathematical Systems Theory* (F. Colonius, U. Helmke, D. Prätzel-Wolters and F. Wirth, Eds.). pp. 255–293. Birkhäuser Verlag.

Mareels, I.M.Y. (1984). A simple selftuning controller for stably invertible systems. *Syst. Control Lett.* **4**, 5–16.

Miller, D.E. and E.J. Davison (1991). An adaptive controller which provides an arbitrarily good transient and steady-state response. *IEEE Trans. Aut. Control* **36**, 68–81.

Morse, A.S. (1983). Recent problems in parameter adaptive control. In: *Outils et Modèles Mathématiques pour l'Automatique, l'Analyse de Systèmes et le Traitment du Signal* (I.D. Landau, Ed.). pp. 733–740. Birkhäuser Verlag.

Ryan, E.P. and C.J. Sangwin (2001). Controlled functional differential equations and adaptive stabilization. *Int. J. Control* **74**, 77–90.

Weiss, G. (1994). Transfer functions of regular linear systems, part 1: characterization of regularity. *Trans. Amer. Math. Soc.* **342**, 827–854.

Willems, J.C. and C.I. Byrnes (1984). Global adaptive stabilization in the absence of information on the sign of the high frequency gain. In: *Lect. Notes in Control and Inf. Sciences 62*. pp. 49–57. Springer-Verlag.