Adaptive tracking within prescribed funnels

Achim Ilchmann, Eugene P. Ryan, Stephan Trenn

***Abstract*— Output tracking of a reference signal (an absolutely continuous bounded function with essentially bounded derivative) is considered in a context of a class of nonlinear systems described by functional differential equations. The primary control objective is tracking with prescribed accuracy: given $\lambda > 0$ (arbitrarily small), ensure that, for every admissible system and reference signal, the tracking error e is ultimately smaller than λ (that is, $\|e(t)\| < \lambda$ for all t sufficiently large). The second objective is guaranteed transient performance: the evolution of the tracking error should be contained in a prescribed performance funnel $\mathcal{F}$. Adopting the simple feedback control structure $u(t) = -k(t)e(t)$, it is shown that the above objectives can be achieved if the gain $k(t) = K_{\mathcal{F}}(t, e(t))$ is generated by any continuous function $K_{\mathcal{F}}$ exhibiting two specific properties formulated in terms of the distance of $e(t)$ to the funnel boundary.**

I. INTRODUCTION

In a precursor [1] to the present paper, a proportional output feedback controller has been introduced that guarantees prespecified tracking behaviour for a class of nonlinear systems described by functional differential equations of the form

$$\dot{y}(t) = f\big(p(t), (Ty)(t), u(t)\big), \quad y_{[-h,0]} = y^0,$$

where, loosely speaking, the parameter $h \geq 0$ quantifies system "memory", p may be thought of as a (bounded) disturbance term, and T is a nonlinear causal operator, for details see Section II. For the underlying system class and

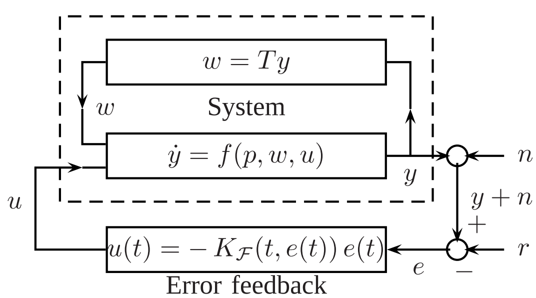

Fig. 1. Universal error feedback control.

any reference signal in the space $W^{1,\infty}$ of locally absolutely continuous bounded functions $r \in L^\infty$ with essentially bounded derivative $\dot{r} \in L^\infty$, the problem of tracking with prescribed asymptotic accuracy and prescribed transient behaviour was formulated in terms of a performance funnel $\mathcal{F}$ (given by the graph of a suitably chosen set-valued map $t \mapsto F(t)$).

The goal is a control structure which, for every admissible system and reference signal, ensures that the graph of the tracking error $e(\cdot)$ is contained in the funnel $\mathcal{F}$. In [1], this goal was achieved by the simple control structure $u(t) = -k(t)e(t)$ with the gain generated by a feedback law of the form $k(t) = K_{\mathcal{F}}(t, e(t))$, where $K_{\mathcal{F}}$ is a continuous function such that, loosely speaking, the reciprocal $1/K_{\mathcal{F}}(t, e)$ provides a particular measure of distance of (t, e) from the boundary of the funnel $\mathcal{F}$ (with the effect that, if the error approaches the boundary, then the gain increases which, in conjunction with a high-gain property of the underlying system class, precludes contact with the boundary).

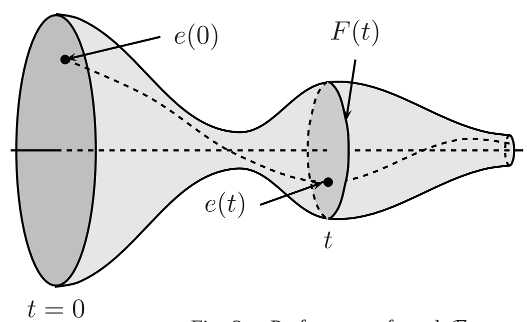

Fig. 2. Performance funnel $\mathcal{F}$.

In [1], the choice of feasible gains includes the scaled (scale factor $1/\varphi$) vertical distance to the funnel

$$K_{\mathcal{F}}(t, e(t)) \;=\; \frac{1/\varphi(t)}{1/\varphi(t) - \|e(t)\|}, \tag{1}$$

where $\varphi \in W^{1,\infty}$ and its reciprocal $1/\varphi(t)$ specifies the radius of the ball $F(t)$ ($\mathcal{F} = \mathrm{graph}(F)$), see Figure 2.

The purpose of the present paper, *vis à vis* its precursor [1], is to extend the class of admissible gain functions $K_{\mathcal{F}}$ by determining structural assumptions on the gain function, which allow for great flexibility in the choice of measure of the distance to the funnel boundary (flexibility which, for example, permits the control to anticipate the future shape of the funnel and to adjust the current control gain accordingly), and which may be of relevance in certain applications. These general results encompass such examples as the unscaled vertical distance (see Figure 3) to the funnel, viz.

$$K_{\mathcal{F}}(t, e(t)) \;=\; \mathrm{dist}(e(t), \partial F(t)) = \frac{1}{1/\varphi(t) - \|e(t)\|},$$

A. Ilchmann and S. Trenn are with the Institute of Mathematics, Technical University Ilmenau, Weimarer Straße 25, 98693 Ilmenau, FRG, ilchmann@mathematik.tu-ilmenau.de, stephan.trenn@stud.tu-ilmenau.de

E.P. Ryan is with the Department of Mathematical Sciences, University of Bath, Claverton Down, Bath BA2 7AY, UK; epr@maths.bath.ac.uk

(wherein $\partial F(t)$ denotes the boundary of the set $F(t)$) or gains $K_{\mathcal{F}}$ based on the future distance (see Figure 4) to the funnel

$$d_f(t, e(t)) \ := \ \inf_{\tau > t} \sqrt{(\tau - t)^2 + \big(\text{dist}(e(t), \partial F(\tau)\big)^2}.$$

Furthermore, we investigate gains based on a numerical future distance (a numerical approximation of the above future distance), and "direction-dependent" gains associated with non-axially-symmetric funnels.

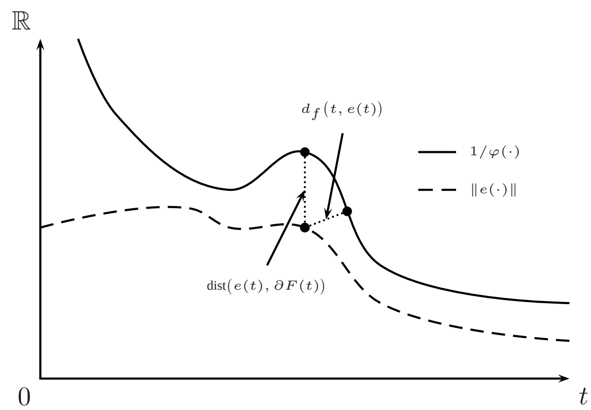

Fig. 3. The distance $d_f(t, e(t))$ to the future funnel, and the unscaled vertical distance dist$(e(t), \partial F(t))$ to the funnel.

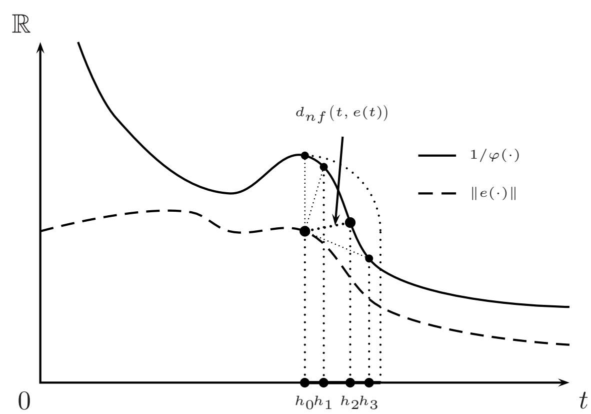

Fig. 4. The numerical distance d_{nf} to the future funnel.

The control strategy investigated in [1] and the present paper, is essentially applicable to the same system class also studied for high-gain adaptive control. Roughly speaking, the system class encompasses relative degree one systems with "weakly stable" zero dynamics and known sign of the high-frequency gain. The main difference to adaptive control strategies (see [2] and the reference therein) is that in the present paper we (i) obey prespecified transient behaviour, (ii) the gain $t \mapsto k(t)$ is not a monotonically non-decreasing function, (iii) the gain is not tuned by a dynamical system (e.g. $\dot{k} = \|e\|^2$ in the adaptive context) and hence may not even be called adaptive, and (iv) no bounds on the nonlinearities of the system need to be known.

[3] have introduced a controller which guarantees prespecified transient behaviour. However, their controller is adaptive with monotonically non-decreasing gain, invokes a piecewise constant switching strategy.

The proposed controller also tolerates *output measurement disturbance* n, provided that the disturbance belongs to the same function class as the reference signals. With reference to Figure 1, the disturbed error signal is then $e = (y + n) - r = y - (r - n)$. Therefore, from a strictly analytical viewpoint, in the presence of output disturbances of class $W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^M)$, the disturbance-free analysis is immediately applicable on replacing the reference signal r by the signal $r - n$. Even though the reference signal r and disturbance signal n are assumed to be of the same class, practically, these signals might be distinguished by their respective spectra (n typically having "high-frequency" content). Moreover, from a practical viewpoint, one might reasonably expect that the disturbance n is "small". For example, if an upper bound $\varepsilon > 0$ of the magnitude of the disturbance is known, viz. $\|n\|_\infty \leq \varepsilon$, and $\lambda > 0$ is the prescribed measure of asymptotic tracking accuracy (for the disturbance free case), then the actual tracking accuracy achieved in the presence of disturbance is quantified by $\hat{\lambda} = \lambda + \varepsilon$. For simplicity of presentation, we consider only the disturbance-free case in the analysis.

The paper is organised as follows. In Section II, we make precise the underlying system class. The control problem is formulated in Section III, wherein the class of reference signals and the performance funnel are described. Section IV elucidates the proposed output feedback control and, in the main result (Theorem 1), establishes the requisite transient and asymptotic behaviour of the closed-loop system. Finally, in Section V, the flexibility in the choice of gain functions $K_{\mathcal{F}}$, alluded to above, is illustrated via diverse examples determined by a variety of measures of distance to the funnel boundary. Owing to page restrictions on this conference paper, all proofs are omitted.

We close the present section with some remarks on notation.

Define $\mathbb{R}_{\geq 0} := [0, \infty)$, $\mathbb{R}_{>0} := (0, \infty)$, $\|x\| := \sqrt{x^T x}$, $x \in \mathbb{R}^n$, and

dist(x,A) $:= \inf_{a \in A} \|x - a\|$, the Euclidean distance of $x \in \mathbb{R}^n$ from a non-empty set $A \subset \mathbb{R}^n$, dist$(\cdot, A)$ is Lipschitz with constant 1,

$\mathbb{B}_\delta(\xi)$ $:= \big\{x \in \mathbb{R}^n \big| \ \|x - \xi\| < \delta\big\}$, the open ball of radius $\delta > 0$ centred at $\xi \in \mathbb{R}^n$,

$C(S;\mathbb{R}^n)$	the set of continuous functions $S \to \mathbb{R}^n$,
$L^\infty(I;\mathbb{R}^n)$	the space of measurable essentially bounded functions $I \to \mathbb{R}^n$, $I \subset \mathbb{R}$ an interval, with norm
$\|x\|_\infty$	$:= \operatorname{ess\,sup}_{t\in I} \|x(t)\|$,
$L^\infty_{\text{loc}}(I;\mathbb{R}^n)$	the space of measurable, locally essentially bounded functions $I \to \mathbb{R}^n$, $I \subset \mathbb{R}$ an interval.
$W^{1,\infty}(\mathbb{R}_{\geq 0};\mathbb{R}^M)$	the set of bounded locally absolutely continuous functions $r : \mathbb{R}_{\geq 0} \to \mathbb{R}^M$ with essentially bounded derivative and norm
$\|x\|_{1,\infty}$	$:= \|x\|_\infty + \|\dot{x}\|_\infty$.

II. System class Σ

Consider the class Σ of infinite-dimensional, nonlinear, M-input u, M-output y systems (p, f, T), given by a controlled nonlinear functional differential equation of the form

$$\dot{y}(t) = f\big(p(t), (Ty)(t), u(t)\big), \quad y_{[-h,0]} = y^0 \qquad (2)$$

with $h \geq 0$, $y^0 \in C\big([-h, 0]; \mathbb{R}^M\big)$, and satisfying the following properties for some $P, Q \in \mathbb{N}$:

(i) $p \in L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^P)$;

(ii) $f \in C\big(\mathbb{R}^P \times \mathbb{R}^Q \times \mathbb{R}^M; \mathbb{R}^M\big)$;

(iii) for every non-empty compact subset $\mathcal{C} \subseteq \mathbb{R}^P \times \mathbb{R}^Q$ and every sequence (u_n) in $\mathbb{R}^M \backslash \{0\}$ the following property (akin to radial unboundedness or weak coercivity) holds:

$$\|u_n\| \to \infty \ \text{ as } \ n \to \infty \qquad \Longrightarrow$$
$$\lim_{n\to\infty} \min_{(v,w)\in\mathcal{C}} \langle u_n, f(v, w, u_n)\rangle / \|u_n\| = \infty\,;$$

(iv) $T : C([-h,\infty); \mathbb{R}^M) \to L^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}^Q)$ denotes an operator of class $\mathcal{T}$, i.e.

a) $\forall\, \delta > 0\ \exists\, \Delta > 0\ \forall\, x \in C\big([-h,\infty); \mathbb{R}^M\big)$:

$\|x\|_\infty \leq \delta \Rightarrow \big\|(Tx)(t)\big\| \leq \Delta$ for a.a. $t \geq 0$;

b) $\forall\, t \geq 0\ \forall x, \xi \in C\big([-h,\infty); \mathbb{R}^M\big)$:

$x|_{[-h,t]} = \xi|_{[-h,t]} \implies$

$(Tx)(s) = (T\xi)(s)$ for a.a. $s \in [0, t]\Big]$;

c) $\forall t \geq 0\, \forall \zeta \in C\big([-h, t]; \mathbb{R}^M\big)\, \exists\, \tau, \delta, c > 0$ $\forall\, x, \xi \in C\big([-h,\infty); \mathbb{R}^M\big)$ with $x|_{[-h,t]} = \zeta = \xi|_{[-h,t]}$ and $x(s), \xi(s) \in \mathbb{B}_\delta(\zeta(t)) \forall s \in [t, t+\tau]$: $\text{ess-sup}_{s\in[t,t+\tau]} \|(Tx)(s) - (T\xi)(s)\| \leq c\, \sup_{s\in[t,t+\tau]} \|x(s) - \xi(s)\|$.

The function p in (2) may be thought of as a (bounded) disturbance term; the non-negative constant h quantifies the "memory" of the system.

Property (iii) generalizes the positive "high-frequency gain" concept in linear systems and, in particular, that (2) has strict relative degree one.

Property (iv)(a) is a crucial "bounded-input, bounded-output" assumption on the operator T.

Property (iv)(b) is an assumption of causality; and Property 4c is a technical assumption on T of a "locally Lipschitz" nature.

Numerous examples can be found in [1], [4] and, furthermore, diverse phenomena are incorporated within the class including, for example, diffusion processes, delays (both point and distributed) and hysteretic effects. The prototypical example is the class of finite-dimensional, linear, minimum-phase systems of relative degree one described by

$$\begin{aligned} \dot{y}(t) &= A_1 y(t) + A_2 z(t) + CB\, u(t), \quad & y(0) &= y^0, \\ \dot{z}(t) &= A_3 y(t) + A_4 z(t), & z(0) &= z^0, \end{aligned}$$

with real matrices of conforming formats, and $(CB)^T + CB > 0$, $\sigma(A_4) \subset \mathbb{C}_-$. We may rewrite the above system in terms of (2) by

$$\begin{aligned} &\dot{y}(t) = A_2 \exp(A_4 t) z^0 + (Ty)(t) + CB\, u(t),\ y(0) = y^0 \\ &(Ty)(t) := A_1 y(t) + A_2 \int_0^t \exp(A_4(t-s)) A_3 y(s) ds. \end{aligned}$$

III. Problem formulation

A. The performance funnel

Let Φ denote the class of functions $\varphi \in W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R})$ which are positive-valued on $(0,\infty)$ and bounded away from zero "at infinity", i.e.,

$$\Phi := \left\{ \varphi \in W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}) \;\middle|\; \begin{array}{l} \varphi(s) > 0\, \forall\, s > 0 \\ \liminf_{s\to\infty} \varphi(s) \in (0,\infty). \end{array} \right\}$$

With $\varphi \in \Phi$, we associate a set-valued map (defined on $\mathbb{R}_{\geq 0}$)

$$F:\ t \mapsto F(t) := \big\{ e \in \mathbb{R}^M \ \big|\ \ \varphi(t)\|e\| < 1 \big\},$$

the graph of which we refer to as the performance funnel

$$\mathcal{F} := \text{graph}(F) := \big\{ (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^M \ \big|\ \ e \in F(t) \big\}.$$

Observe that (i) $\varphi(0) = 0$ is permissible, in which case, $F(0) = \mathbb{R}^M$, and (ii) for every $\varphi \in \Phi$ and $\tau > 0$, there exists $\mu > 0$ such that $\varphi(t) \geq \mu$ for all $t \geq \tau$, and so $F(t) \subset \mathbb{B}_{1/\mu}(0)$ for all $t \geq \tau$.

As a concrete example, for $\lambda > 0$, $\tau > 0$ and $\varepsilon \in (0, 1)$, the choice

$$t \mapsto \varphi(t) = \frac{t}{([1-\varepsilon]t + \varepsilon\tau)\lambda}$$

yields an associated performance funnel $\mathcal{F}$ which reflects an overall objective of attaining tracking accuracy λ in prescribed time τ.

B. Class of reference signals and control objective

As reference signals r, we allow bounded locally absolutely continuous functions with bounded derivative, i.e. $r \in W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^M)$ with norm given by $\|r\|_{1,\infty} := \|r\|_\infty + \|\dot{r}\|_\infty$.

Given $\varphi \in \Phi$ and its associated performance funnel $\mathcal{F}$, the control objective is a single feedback strategy ensuring that, for each reference signal $r \in W^{1,\infty}$ and every system of class Σ, the tracking error $e = y - r$ has graph in $\mathcal{F}$ (equivalently: $e(t) \in F(t)$ for all $t \geq 0$), and all variables are bounded.

IV. Output feedback control

Let $\varphi \in \Phi$ determine a performance funnel $\mathcal{F}$ and let $r \in W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^M)$. We seek to achieve the above control objective via the simple proportional time-varying output error feedback

$$u(t) = -k(t)e(t), \quad k(t) = K_{\mathcal{F}}(t, e(t)), \tag{3}$$

where $e(t) = y(t) - r(t)$, whilst ensuring boundedness of the gain k. Here, $K_{\mathcal{F}} : \mathcal{F} \to \mathbb{R}_{\geq 0}$ is a continuous function chosen to ensure the intuition underlying the control approach: $K_{\mathcal{F}}$ is such that, if $(t, e(t))$ approached the boundary of the funnel $\mathcal{F}$, then the gain $k(t) = K_{\mathcal{F}}(t, e(t))$ increases at a rate sufficient to preclude – via an implicit high-gain stability property of underlying system class Σ – boundary contact, thereby maintaining the error evolution within the performance funnel. Next, we elucidate two properties which, when imposed on the gain function $K_{\mathcal{F}}$, confirm this intuition.

A. Requisite properties of the gain function

Let $\varphi \in \Phi$, with associated map $t \mapsto F(t)$ and performance funnel $\mathcal{F} = \text{graph}(F)$. For each $t \in \mathbb{R}_{\geq 0}$, we denote the boundary of the set $F(t)$ by $\partial F(t)$. Let $K_{\mathcal{F}} : \mathcal{F} \to \mathbb{R}_{\geq 0}$ be a continuous function. We impose only the following additional properties on $K_{\mathcal{F}}$.

Property A: $\forall\, K > 0 \;\; \exists\, \varepsilon > 0 \;\; \forall\, (t, e) \in \mathcal{F}$:

$$\big[\; \text{dist}(e, \partial F(t)) \leq \varepsilon \quad \Rightarrow \quad K_{\mathcal{F}}(t, e) \geq K \;\big].$$

Property B: $\forall \varepsilon > 0 \; \forall \delta > 0 \; \exists K > 0 \;\; \forall (t, e) \in \mathcal{F}$:

$$\big[\text{dist}(e, \partial F(t)) \geq \varepsilon \;\; \wedge \;\; t \geq \delta \Rightarrow K_{\mathcal{F}}(t, e) \leq K\big].$$

The essence of these properties is as follows. Property A ensures that, in (3), if the tracking error $e(t)$ is close to the funnel boundary, then the associated gain value $k(t)$ is large. Property B, loosely speaking, obviates the need for large gain values away from the funnel boundary.

B. The main result

Theorem 1: Let $(f, p, T) \in \Sigma$. Let $\varphi \in \Phi$ with associated map F and performance funnel $\mathcal{F} = \text{graph}(F)$. Let $K_{\mathcal{F}} : \mathcal{F} \to \mathbb{R}_{\geq 0}$ be continuous with Properties A and B.
For any reference signal $r \in W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^M)$ and initial data $y^0 \in C\big([-h, 0]; \mathbb{R}^M\big)$ such that $y^0(0) - r(0) \in F(0)$, there exists a solution of the closed-loop initial-value problem (2), (3), that is,

$$\begin{aligned} &\dot{y}(t) = f\big(p(t), (Ty)(t), -K_{\mathcal{F}}(t, e(t))\, e(t)\big), \\ &e(t) = y(t) - r(t) \in F(t), \quad y|_{[-h,0]} = y^0 . \end{aligned}$$

Every solution can be extended to a maximal extension $y : [-h, \omega) \to \mathbb{R}^n$ and every maximal solution has the following properties

(i) $\omega = \infty$,
(ii) $t \mapsto k(t) = K_{\mathcal{F}}(t, y(t) - r(t))$ is bounded,
(iii) there exists $\varepsilon > 0$ such that, for all $t \in [0, \infty)$, $\text{dist}\big(y(t) - r(t), \partial F(t)\big) \geq \varepsilon$.

V. Gain functions

In this section we describe various choices of continuous gain function $K_{\mathcal{F}}$, with the requisite Properties A and B, which are feasible for the feedback (3).

A. Scaled vertical distance

Here, we base the gain function on measurements of the distance of the instantaneous error $e(t)$ from the boundary of the set $F(t)$: this approach uses only funnel information at current time t and, in particular, does not anticipate the future shape of the funnel boundary.

With reference to Figure 3, for $(t, e) \in \mathcal{F}$, we refer to the distance $\text{dist}(e, \partial F(t)) = 1/\varphi(t) - \|e\|$ (with the convention that $\text{dist}(e, \partial F(0)) = \infty$ if $\varphi(0) = 0$) as the vertical distance from (t, e) to the funnel boundary: in incorporating this distance in the design of gain functions $K_{\mathcal{F}}$, we allow for scaling by a suitable function ψ and refer to the quantity $\psi(t)\text{dist}(e, \partial F(t))$ as a scaled vertical distance.

Proposition 2: Let $\varphi, \psi \in \Phi$ such that $\lim_{t \to 0+} \psi(t)\, \varphi(t)^{-1} =: \psi_0 \in (0, \infty]$, and let $\mathcal{F}$ be the performance funnel associated with φ. Assume that $\beta : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ is continuous, unbounded and non-increasing. Then

$$K_{\mathcal{F}} : \mathcal{F} \to \mathbb{R}_{\geq 0}, \; (t, e) \mapsto$$

$$\begin{cases} \beta\big(\psi(t)\, \text{dist}(e, \partial F(t))\big), & t > 0 \\ \beta\big(\psi_0 - \psi(0)\|e\|\big), & t = 0 \text{ and } \psi_0 < \infty \\ \beta_* := \lim_{s \to \infty} \beta(s), & t = 0 \text{ and } \psi_0 = \infty \end{cases}$$

is continuous and has Properties A and B.

It can be shown, that the strategy introduced in [1] is also covered by a function $K_{\mathcal{F}}$ satisfying Properties 1 and 2.

The simplest example, covered by Proposition 2, is the unscaled vertical distance: for $\psi \equiv 1$ and $\beta : s \mapsto 1/s$, we have, for all $(t, e) \in \mathcal{F}$,

$$\begin{aligned} K_{\mathcal{F}}(t, e) &= \tfrac{1}{\text{dist}(e, \partial F(t))} \\ &= \begin{cases} 0, & t = \varphi(0) = 0, \\ \Big(\tfrac{1}{\varphi(t)} - \|e\|\Big)^{-1}, & \text{otherwise}. \end{cases} \end{aligned}$$

B. The distance to the future funnel

As already mentioned, the scaled vertical distance, investigated in the previous sub-section, uses only instantaneous funnel information. It is of theoretical interest, and also of relevance in certain applications, to incorporate anticipation of the future funnel shape in determining the current gain value. To this end, we next investigate the adoption of the distance $d_f(t, e)$ of $(t, e) \in \mathcal{F}$ to the future funnel boundary in the design of gain functions $K_{\mathcal{F}}$ with Properties A and B. For $\varphi \in \Phi$, with associated map F and performance funnel $\mathcal{F}$, this distance is defined for all $(t,e) \in \mathcal{F}$, with reference to Figure 3, as follows

$$d_f(t,e) := \inf_{\tau > t} \sqrt{(\tau - t)^2 + \big(\text{dist}(e, \partial F(\tau)\big)^2}.$$

In contrast with the (scaled) vertical distance of the previous subsection (which is infinite at $(0,e)$ in cases where $\varphi(0) = 0$), the distance $d_f(t,e)$ is finite for all $(t,e) \in \mathcal{F}$.

Proposition 3: Let $\varphi \in \Phi$, with associated map F and performance funnel $\mathcal{F}$. Then the function $d_f : \mathcal{F} \to \mathbb{R}_{>0}$ is continuous.
Let, furthermore, $\psi \in \Phi$ be such that $\psi(0) > 0$ and assume that $\beta : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ is continuous, unbounded and non-increasing. Then

$$K_{\mathcal{F}} : \mathcal{F} \to \mathbb{R}_{\geq 0}, \quad (t,e) \mapsto \beta\big(\psi(t) d_f(t,e)\big)$$

is continuous and has Properties A and B.

C. A numerical future distance

The following distance is less sensitive to the change of the funnel boundary but easier to calculate. Choose, for $N \in \mathbb{N}$, the partition

$$0 = h_0 \ < \ h_1 \ < \ \ldots \ < \ h_N \leq 1.$$

Let $\varphi \in \Phi$ such that $\varphi(0) > 0$, and let $\mathcal{F}$ be the associated performance funnel. Define for all $(t,e) \in \mathcal{F}$

$$d(t,e) := \text{dist}(e, \partial F(t)) < \infty$$

and the numerical future distance, with reference to Figure 4, as

$$\begin{aligned} & d_{nf}(t,e) := \\ & \min_{0 \leq i \leq N} \text{dist}\big((t, \|e\|), (t + h_i d(t,e), 1/\varphi(t + h_i d(t,e))\big) \\ & = \min_{0 \leq i \leq N} \sqrt{\Big(h_i d(t,e)\Big)^2 + \Big(\tfrac{1}{\varphi(t + h_i d(t,e))} - \|e\|\Big)^2}. \end{aligned}$$

The numerical future distance calculates, at any time t, the distance to the funnel boundary at finitely many future points $t + h_i d(t,e)$. Since $\text{dist}\big((t, \|e\|), (t+\delta, 1/\varphi(t+\delta)\big) \geq \delta$ for all $\delta > 0$, it is not necessary to look further into the future than the value of the actual "vertical" distance $\text{dist}(e, \partial F(t)) = d(t,e)$. Note that, since $h_0 = 0$, the inequality

$$\begin{aligned} & d_{nf}(t,e) \\ & \leq \text{dist}\big((t, |e|), (t + h_0 d(t,e), 1/\varphi(t + h_0 d(t,e))\big) \\ & = \text{dist}(e, \partial F(t)) \quad \forall\, (t,e) \in \mathcal{F} \end{aligned}$$

implies that any future point with a time-distance greater than $\text{dist}(e, \partial F(t)) = d(t,e)$ has no influence on $d_{nf}(t,e)$.

Proposition 4: Let φ, $\psi \in \Phi$ with $\varphi(0) > 0$ and $\psi(0) > 0$, and let $\mathcal{F}$ be the performance funnel associated with φ. Assume that $\beta : (0, \infty) \to \mathbb{R}_{\geq 0}$ is a continuous, non-increasing and unbounded function. Then

$$K_{\mathcal{F}} : \mathcal{F} \to \mathbb{R}_{\geq 0}, \quad (t,e) \mapsto \beta\big(\psi(t) d_{nf}(t,e)\big)$$

is continuous and satisfies the Properties A and B in Sub-section IV-A.

D. A direction-depending gain

All gains $K_{\mathcal{F}}$ of the previous Sub-sections V-A-V-C depend only on the norm of the error. Now a gain is introduced which allows a scaling depending on the direction $e/\|e\|$ by the continuous function

$$s \in C(\mathbb{S}^{M-1}; \mathbb{R}_{>0}).$$

Proposition 5: Let $\varphi \in \Phi$ with associated performance funnel $\mathcal{F}$, and $\hat{K}_{\mathcal{F}}$ denote any of the the gain functions in Sub-sections V-A-V-C. Then $K_{\mathcal{F}}$ defined on $\mathcal{F}$ by

$$K_{\mathcal{F}}(t,e) := \begin{cases} s(e/\|e\|)\, \|e\|\, \hat{K}_{\mathcal{F}}(t,e), & e \neq 0 \\ 0, & e = 0 \end{cases}$$

is continuous and satisfies Properties A and B in Sub-section IV-A.

VI. Conclusions

We have studied an output feedback law $u(t) = -k(t)e(t)$ which ensures tracking with prespecified accuracy and, more importantly, guarantees transient behaviour of the evolution of the tracking error within a prescribed performance funnel. The feedback law is simple in its design: the gain $k(t) = K_{\mathcal{F}}(t, e(t))$ depends on time t and error $e(t)$ where, loosely speaking, the reciprocal $1/K_{\mathcal{F}}(t,e)$ provides a particular measure of the distance of (t,e) from the boundary of the funnel $\mathcal{F}$. The effect is that, if the error approaches the boundary, then the gain increases which, in conjunction with a high-gain property of the underlying system class, precludes contact with the boundary.

Compared to ubiquitous high-gain adaptive control strategies (which apply to the same class of nonlinear systems) it may be surprising that the gain is not a monotone function and, most importantly, the feedback law ensures a prespecified transient behaviour.

The main result of the present note is a feedback law which allows for a great flexibility of the measures of the distance to the boundary of the funnel. This permits the control to anticipate the future shape of the funnel and to adjust the current control gain accordingly.

References

[1] A. Ilchmann, E. Ryan, and C. Sangwin, "Tracking with prescribed transient behaviour," *ESAIM Control, Opt. and Calculus of Variations*, vol. 7, pp. 471–493, 2002.

[2] ——, "Systems of controlled functional differential equations and adaptive tracking," *SIAM J. of Control and Optim.*, vol. 40, pp. 1746–1764, 2002.

[3] D. Miller and E. Davison, "An adaptive controller which provides an arbitrarily good transient and steady-state response," *IEEE Trans. Aut. Control*, vol. 36, pp. 68–81, 1991.

[4] E. Ryan and C. Sangwin, "Controlled functional differential equations and adaptive stabilization," *Int. J. Control*, vol. 74, pp. 77–90, 2001.