Differential-Algebraic Inclusions with Maximal Monotone Operators

Kanat Camlibel Luigi Iannelli Aneel Tanwani Stephan Trenn

***Abstract*— The term differential-algebraic inclusions (DAIs) not only describes the dynamical relations using set-valued mappings, but also includes the static algebraic inclusions, and this paper considers the problem of existence of solutions for a class of such dynamical systems described by the inclusion**

$$\frac{\mathrm{d}}{\mathrm{d}t}Px \in -\mathcal{M}(x)$$

for a symmetric positive semi-definite matrix $P \in \mathbb{R}^{n\times n}$, and a maximal monotone operator $\mathcal{M} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. The existence of solutions is proved using the tools from the theory of maximal monotone operators. The class of solutions that we study in the paper have the property that, instead of the whole state, only Px is absolutely continuous and unique. This framework, in particular, is useful for studying passive differential-algebraic equations (DAEs) coupled with maximal monotone relations. Certain class of irregular DAEs are also covered within the proposed general framework. Applications from electrical circuits are included to provide a practical motivation.

I. Introduction

Set-valued dynamical systems are particularly useful for modeling when one does not have complete knowledge of the system parameters, or when the relation between certain variables of the system is not necessarily one-to-one. On other instances, the only information about the variables of the system appears in the form of an inclusion in a static set. To handle such scenarios, this article proposes the study of a relatively new class of differential inclusions, which not only comprise dynamic evolution of the state variable but also includes static algebraic set-valued relations. We use the term differential-algebraic inclusions (DAIs) to describe such systems, and study the inclusions of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}Px \in -\mathcal{M}(x) \tag{1}$$

over a compact interval $[0, T]$, where $x : [0, T] \to \mathbb{R}^n$ denotes the state trajectory of the system, $P \in \mathbb{R}^{n\times n}$ is a symmetric positive **semi-definite** matrix and $\mathcal{M} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a maximal monotone operator (see Definition 1). It is clear that, due to semi-definiteness of the matrix P, there are certain static relations encoded in (1) which make it difficult to study the solutions of system class (1) with existing tools from the literature and, in particular, the theory of differential inclusions with maximal monotone operators cannot be directly applied here.

K. Camlibel is with University of Groningen, The Netherlands. L. Iannelli is with University of Sannio in Benevento, Italy. A. Tanwani is with LAAS–CNRS, University of Toulouse, France. S. Trenn is with University of Kaiserslautern, Germany. Corresponding author: aneel.tanwani@laas.fr

For system class (1), if $P = I$, then the resulting differential inclusion (DI) $\dot{x} \in -\mathcal{M}(x)$ is a well-studied object in the literature [3]. Such DIs form an important class of set-valued and nonsmooth systems. The applications of DIs with maximal monotone operator can be seen in various forms. Certain class of evolution inclusions where the set-valued dynamics are due to subdifferential of the indicator function of a convex set are a particular case [9], which have found useful applications in the modeling of electrical and mechanical systems [1], [17]. Another instance of such DIs is observed in differential variational inequalities [10] which appear in the solutions of the optimal control problems. More recently, the control design problems for such systems has also attracted a lot of attention [14], [15]. The study of DAIs (1) is thus aimed at enlarging the class of systems treated in aforementioned references, and finding applications which cannot be treated within the existing frameworks.

Several physical systems are modeled as ordinary differential equations (ODEs) coupled with maximal monotone relations. In such cases, it often happens that the set-valued mappings that describe the maximal monotone relation are multiplied by non-square matrices which do not render the resulting map maximal monotone. This problem is solved in the literature by imposing a passivity relation between the variables constrained by the maximal monotone relation [2], [5]. This allows for rewriting the entire system dynamics as a DI with maximal monotone relation. Going by this line of thought, one motivation for studying the system class (1) comes from looking at a system of differential-algebraic equations (DAEs) (see [16] for details), where the forcing terms are described by a maximal monotone relation. More formally, we consider DAEs of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}E\,x(t) = Ax(t) + Bz(t) \tag{2a}$$

$$w(t) = Cx(t) + Dz(t), \tag{2b}$$

where the variable $z(\cdot)$ is related to $w(\cdot)$ through the relation

$$w(t) \in \mathcal{F}(-z(t)) \tag{2c}$$

for some given set-valued maximal monotone operator $\mathcal{F} : \mathbb{R}^{d_m} \rightrightarrows \mathbb{R}^{d_m}$. When $E = I$, it was shown in [5] that under passivity assumption on the quadruple (A, B, C, D), the system (1) can be rewritten as a differential inclusion of the form $\dot{x} \in -\mathcal{M}(x)$ for some maximal monotone $\mathcal{M}$. One can then invoke the classical results from [3] to obtain existence and uniqueness of the solutions. Trying to generalize this idea for a singular E matrix does not always lead to a differential inclusion with maximal monotone operators. It is shown in this paper that under appropriate passivity assumptions on the matrix quintuple (E, A, B, C, D), system (2) can be rewritten in the form of inclusion (1), hence the solution theory for system (1) is also very relevant for studying the dynamical system (2).

While system (2) already presents a generalization from the system class studied in [5], the utility of system class (2) can also be found in the modeling of electrical circuits with nonsmooth devices such as diodes and switches. An example is included in Section V for this purpose.

The remainder of the paper is organized as follows: Some basic definitions from convex analysis and the preliminary results are stated in Section II. The main result on existence of solutions for system class (1) appears in Section III. These results are applied to study DAEs with maximal monotone relations (2) under passivity assumption in Section IV, followed by an illustrative example in Section V.

II. Preliminaries and Motivation

In this section, we recall the basic definitions and some standard results which will be used later in the paper.

A. Convex Sets

The results related to analysis of convex sets have been borrowed from [12], [13]. For a set $\mathcal{S} \subset \mathbb{R}^n$, we denote the interior by $\mathrm{int}(\mathcal{S})$, the relative interior by $\mathrm{rint}(\mathcal{S})$, and the closure by $\mathrm{cl}(S)$. The horizon cone of $\mathcal{S}$, denoted $\mathcal{S}_\infty$, is defined as

$$\mathcal{S}_\infty := \{x \,|\, \exists\, x^v \in \mathcal{S}, \lambda^v \searrow 0 \text{ such that } \lambda^v x^v \to x\}.$$

For $\mathcal{S}$ convex, we denote the normal cone to $\mathcal{S}$ at x by $\mathcal{N}_\mathcal{S}(x)$. For a linear map $L : \mathbb{R}^m \to \mathbb{R}^n$, the kernel and range space are denoted by $\ker L$ and $\mathrm{im}\, L$, respectively. Finally, we define $L^{-1}(\mathcal{S}) := \{x \in \mathbb{R}^m \,|\, L(x) \in \mathcal{S}\}$.

B. Maximal Monotone Operators

A set-valued mapping $\mathcal{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called monotone, if

$$\langle y_1 - y_0, x_1 - x_0 \rangle \geq 0, \tag{3}$$

whenever $y_0 \in \mathcal{F}(x_0), y_1 \in \mathcal{F}(x_1)$.

The domain of the mapping $\mathcal{F}$, denoted $\mathrm{dom}(\mathcal{F})$ is the set $\{x \in \mathbb{R}^n \,|\, \mathcal{F}(x) \neq \emptyset\}$.

Definition 1 (Maximal monotonicity): A monotone mapping $\mathcal{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called *maximal monotone* if for every $(\hat{x}, \hat{y}) \in \mathbb{R}^n \times \mathbb{R}^n \backslash \mathrm{graph}(\mathcal{F})$, there exists $(x, y) \in \mathrm{graph}(\mathcal{F})$ such that $\langle y - \hat{y}, x - \hat{x} \rangle < 0$. In other words, $\mathcal{F}$ is maximal monotone if an enlargement of $\mathrm{graph}(\mathcal{F})$ is not possible without destroying monotonicity.

Maximal monotonicity is a very important concept in variational analysis, as one readily sees that the subdifferential of convex (possibly nondifferentiable) functions are maximally monotone [13, Theorem 12.17]. The mappings defined by positive semi-definite matrices are monotone, and continuous, hence they are also maximally monotone [13, Example 12.7]. The following characterization of maximal monotone operators appeared in [8], and will be used in the derivation of certain results in the paper.

Proposition 1: A set-valued mapping $\mathcal{M} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal monotone if, and only if, the following hold

i) $\mathcal{M}$ is monotone,
ii) there exists a convex set $\mathcal{S}_\mathcal{M} \subseteq \mathrm{dom}(\mathcal{M}) \subseteq \mathrm{cl}(\mathcal{S}_\mathcal{M})$,
iii) $\mathcal{M}(x)$ is convex for all $x \in \mathrm{dom}(\mathcal{M})$,
iv) for all $x \in \mathrm{dom}(\mathcal{M})$, $\mathcal{M}(x)_\infty = \mathcal{N}_{\mathrm{cl}(\mathrm{dom}(\mathcal{M}))}(x)$, where, by ii), $\mathrm{cl}(\mathrm{dom}(\mathcal{M}))$ is a convex set,
v) $\mathrm{graph}(\mathcal{M})$ is closed.

As already stated in the introduction, there exist unique solutions to differential inclusions with maximal monotone operators on the right-hand side.

Theorem 1: Consider a maximal monotone mapping $\mathcal{M} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, and the differential inclusion

$$\dot{x} \in -\mathcal{M}(x), \quad x(0) = x_0. \tag{4}$$

For each $x_0 \in \mathrm{dom}(\mathcal{M})$, there exists a unique Lipschitz continuous function $x : [0, \infty) \to \mathbb{R}^n$ such that

- $x(t) \in \mathrm{dom}(\mathcal{M})$ and (4) holds a.e.
- $\dot{x} \in \mathcal{L}^\infty([0, \infty), \mathbb{R}^n)$ and $\|\dot{x}\|_\infty \leq |\mathcal{M}^0(x_0)|$, where $\mathcal{M}^0(x)$ denotes the least norm element of the set $\mathcal{M}(x)$.
- if x^1 and x^2 satisfy (4) with initial conditions $x^1(0) = x_0^1$ and $x^2(0) = x_0^2$, then

$$|x_1(t) - x_2(t)| \leq |x_0^1 - x_0^2|, \quad \forall\, t \geq 0.$$

There are two commonly used techniques, both somewhat constructive, for proving the result of Theorem 1. The first one relies on introducing a sequence of single-valued Lipschitz functions which converge to the least-norm element of the set-valued mapping $\mathcal{M}$ in the limit. The solutions of the resulting differential equations, called the Yosida-Moreau approximations, can then be shown to converge to a function that satisfies the properties listed in Theorem 1.

The second technique relies on constructing a sequence of piecewise constant functions. Each element of this sequence corresponds to a discrete approximation of the inclusion (4) with a certain sampling period. As the sampling period converges to zero, the corresponding sequence is shown to converge to the unique solution of system (4), see [3], or a more recent survey [11], for details on proof techniques.

C. Theoretical Motivation

We now state an academic example comprising a system of the form (2) which can not be represented as $\dot{x} \in -\mathcal{M}(x)$ for a maximal monotone $\mathcal{M}$. For that, consider the system (2) defined by the quintuple

$$(E, A, B, C, D) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & \alpha & 0 \end{bmatrix}, 0 \right) \tag{5}$$

with $-z \in \mathcal{F}^{-1}(w) = \max\{0, w\}$, and some fixed scalar α.

We consider the set-valued operator $\overline{\mathcal{F}}(x) := -Ax + B\mathcal{F}^{-1}(Cx)$, and it is observed that

$$\begin{aligned} x &\mapsto \overline{\mathcal{F}}(x) \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &\mapsto \begin{pmatrix} \max\{0, x_1 + \alpha x_2\} \\ -x_2 + \max\{0, x_1 + \alpha x_2\} \\ -x_3 + \max\{0, x_1 + \alpha x_2\} \end{pmatrix}. \end{aligned}$$

To ensure that $\overline{\mathcal{F}}(x) \in \operatorname{dom}(E^{-1}) = \operatorname{im} E$, we must take $x_3 = \max\{0, x_1 + \alpha x_2\}$. This leads to $\dot{x} \in -\mathcal{M}(x)$ where the set-valued map $\mathcal{M}$ is given by

$$x \mapsto E^{-1}(\overline{\mathcal{F}}(x)), \qquad \overline{\mathcal{F}}(x) \in \operatorname{im} E$$
$$x \mapsto \left\{ \begin{pmatrix} \max\{0, x_1 + \alpha x_2\} \\ \sigma \\ -x_2 + \max\{0, x_1 + \alpha x_2\} \end{pmatrix} \middle| \sigma \in \mathbb{R} \right\}.$$

By choosing $\sigma < x_3 - \frac{2\|Cx\|^2}{x_2}$, one can find $\hat{y} \in E^{-1}(\overline{\mathcal{F}}(\hat{x}))$ such that $\hat{x}^\top \hat{y} < 0$. Thus, the system defined by matrices in (5) cannot be described by inclusions of the form $\dot{x} \in -\mathcal{M}(x)$ for some maximal monotone operator $\mathcal{M}$, and hence the results from existing literature cannot be used to analyze the solutions for the system class (2). For this reason, we are motivated to look at the class of inclusions (1).

III. Solutions of DAIs

The aim of this paper is to show existence of solution for system (1). The presence of some algebraic inequalities require us to define an appropriate concept of solution for such system class.

Definition 2: We call $x : [0, \infty) \to \mathbb{R}^n$ a solution of (1), if it satisfies the following two properties:

- For every $t \geq 0$, we have $x(t) \in \mathcal{M}^{-1}(\operatorname{im} P)$,
- Px is Lipschitz continuous, and the inclusion (1) holds for Lebesgue almost every $t \geq 0$.

Based on this definition, we now state the first main result of this paper on existence of solutions for the DAI (1).

Theorem 2: Consider system (1) and assume that P is a symmetric positive semidefinite matrix, and $\mathcal{M}$ is a maximal monotone operator. Assume that

$$\operatorname{rint}(\operatorname{im} \mathcal{M}) \cap \operatorname{im} P \neq \emptyset. \tag{6}$$

Then, for each $x_0 \in \mathcal{M}^{-1}(\operatorname{im} P)$, there exists a solution x with $x(0) = x_0$ in the sense of Definition 2 such that

- $\frac{\mathrm{d}}{\mathrm{d}t} Px \in \mathcal{L}^\infty([0, \infty), \mathbb{R}^n)$, and $\|\frac{\mathrm{d}}{\mathrm{d}t} Px\|_\infty \leq c^0(x(0))$, where $c^0(x(0))$ is a constant that depends on the initial condition.
- if x^1 and x^2 are two solutions to (1), then
$$|Px^1(t) - Px^2(t)| \leq \gamma_P |x^1(0) - x^2(0)|$$
for some constant $\gamma_P > 0$. In particular, Px is unique.
- there exists a matrix $Q \in \mathbb{R}^{(n - \operatorname{rank} P) \times n}$, and a time-varying convex-valued mapping[1] $\mathcal{S} : [0, \infty) \rightrightarrows \mathbb{R}^{n - \operatorname{rank} P}$ such that for $t \geq 0$:
$$\ker Q = \operatorname{im} P, \quad \text{and} \quad Qx(t) \in \mathcal{S}(t).$$

The above theorem states that, under condition (6), solutions of (1) starting from an admissible set are defined for all times. However, the solutions are not necessarily unique. It is observed that, a certain part of the solution described by Px is sufficiently regular and uniquely determined. The rest of the solution, which is not necessarily unique, is described by Qx. At this moment, we only introduce the time-varying set $\mathcal{S}(t)$ to which this nonunique component belongs and essentially it comes from solving the static algebraic inclusion in (1). Whether it is possible to make a selection from the multivalued component $\mathcal{S}(\cdot)$ which satisfies some regularity conditions, is still a topic under investigation. For the proof of Theorem 2, we first state two intermediate lemmas.

[1]The exact construction of $\mathcal{S}(\cdot)$ is a part of the proof of Theorem 2.

Lemma 1: Let $P = P^\top \geq 0$ and let $\mathcal{M} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a maximal monotone operator. Then there exists $\widehat{\mathcal{M}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ maximal monotone and a coordinate transformation $\xi = Hx$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} Px \in -\mathcal{M}(x) \iff \xi = Hx \ \wedge \ \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \xi \in -\widehat{\mathcal{M}}(\xi).$$

Proof: Since P is symmetric, it is diagonalizable with an orthogonal coordinate transformation, i.e. there is an invertible matrix V with $V^{-1} = V^\top$ and a diagonal full rank matrix P_1 such that

$$P = V^\top \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} V = V^\top \begin{bmatrix} P_1^{1/2} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_1^{1/2} & 0 \\ 0 & I \end{bmatrix} V.$$

Let us now define the matrix H and the operator $\widehat{\mathcal{M}}$ as

$$H := \begin{bmatrix} P_1^{1/2} & 0 \\ 0 & I \end{bmatrix} V \quad \text{and} \quad \widehat{\mathcal{M}}(\xi) := H^{-\top} \mathcal{M}(H^{-1}\xi)$$

then $\ker P = \ker \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] H$. Hence, $\frac{\mathrm{d}}{\mathrm{d}t} Px$ exists if and only if $\frac{\mathrm{d}}{\mathrm{d}t} \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] Hx$ exists, which results in

$$\frac{\mathrm{d}}{\mathrm{d}t} Px \in -\mathcal{M}(x) \iff \xi = Hx \ \wedge \ \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \xi \in -\widehat{\mathcal{M}}(\xi).$$

To show that $\widehat{\mathcal{M}}$ is maximal monotone, we make use of the characterization given in Proposition 1. Using the fact that $\mathcal{M}$ being maximal monotone already satisfies the five listed properties, it is now shown that the same holds for $\widehat{\mathcal{M}}$.

i) To check monotonicity, take $z_i \in H(\operatorname{dom} \mathcal{M})$ and $y_i \in \widehat{M}(z_i)$, for $i = 1, 2$, then $(H^\top y_i, H^{-1} z_i) \in \operatorname{graph}(\mathcal{M})$, and it follows that
$$\langle y_1 - y_2, z_1 - z_2 \rangle = \left\langle H^\top (y_1 - y_2), H^{-1}(z_1 - z_2) \right\rangle \geq 0.$$
ii) It is noted that $\operatorname{dom} \widehat{\mathcal{M}} = H(\operatorname{dom} \mathcal{M})$, and the desired inclusions follow by taking $\mathcal{S}_{\widehat{\mathcal{M}}} = H \mathcal{S}_{\mathcal{M}}$.
iii) For each $z \in H \operatorname{dom} \mathcal{M}$, $\mathcal{M}(H^{-1} z)$ is convex, and so is $H^{-\top} \mathcal{M}(H^{-1} z) = \widehat{\mathcal{M}}(z)$.
iv) Since $\operatorname{cl}(\operatorname{dom} \mathcal{M})$ is convex, then so is $\operatorname{cl}(\operatorname{dom} \widehat{\mathcal{M}}) = \operatorname{cl}(H \operatorname{dom} \mathcal{M})$. Moreover,
$$\begin{aligned}(\widehat{\mathcal{M}}(z))_\infty &= H^{-\top} (\mathcal{M}(H^{-1} z))_\infty \\ &= H^{-\top} \mathcal{N}_{\operatorname{cl}(\operatorname{dom}(\mathcal{M}))}(H^{-1} z) \\ &= \mathcal{N}_{\operatorname{cl}(H \operatorname{dom}(\mathcal{M}))}(z).\end{aligned}$$
v) Using the fact that H^{-1} is single-valued and linear, it can be shown that $\operatorname{graph}(\widehat{\mathcal{M}})$ is closed.

This completes the proof. ∎

Lemma 2: Consider a maximal monotone mapping $\widehat{\mathcal{M}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, and decompose $\xi \in \mathbb{R}^n$ as $\xi := (\xi_1^\top, \xi_2^\top)^\top$ with $\xi_i \in \mathbb{R}^{n_i}$, $i = 1, 2$. Assume that $\operatorname{rint}(\operatorname{im} \widehat{\mathcal{M}}) \cap (\mathbb{R}^{n_1} \times$

$\{0\}^{n_2}) \neq \emptyset$. Then, for each $\xi(0) \in \mathcal{S}_{\text{cons}} := \widehat{\mathcal{M}}^{-1}(\mathbb{R}^{n_1} \times \{0\}^{n_2})$, the following differential inclusion

$$\begin{pmatrix} \dot{\xi}_1 \\ 0 \end{pmatrix} \in -\widehat{\mathcal{M}}(\xi) \tag{7}$$

has a solution $\xi : [0,\infty) \to \mathbb{R}^n$, with $\xi(t) \in \mathcal{S}_{\text{cons}}$ for each $t \geq 0$, such that

- ξ_1 is unique and Lipschitz continuous,
- $\dot{\xi}_1 \in \mathcal{L}^\infty([0,\infty), \mathbb{R}^{n_1})$, and $|\dot{\xi}_1| \leq c^1(\xi(0))$, for some constant c^1 depending on $\xi_1(0)$.
- $\xi_2(t) \in \widehat{\mathcal{S}}(t)$, where $\widehat{\mathcal{S}} : [0,\infty) \rightrightarrows \mathbb{R}^{n_2}$ is a convex-valued operator.

Remark 1: Roughly speaking the statement of Lemma 2 says that the solution of (7) has some regularity associated to the ξ_1 component, while ξ_2 may not even be single-valued. In the proof of this lemma, we express ξ_1 as a solution to a reduced order differential inclusion with a maximal monotone operator which guarantees the uniqueness, regularity, and the bounds on its derivatives. For ξ_2, we are able to compute a time-varying set-valued mapping $\widehat{\mathcal{S}}$, such that any selection of $\xi_2(t) \in \widehat{\mathcal{S}}(t)$ would constitute a solution. It is not clear at this point whether $\widehat{\mathcal{S}}$ satisfies some continuity condition (in the sense of set-valued mappings), so that appropriate regularity claims can be made for ξ_2.

Proof of Lemma 2. Let $\mathcal{G}(y) := \widehat{\mathcal{M}}^{-1}(y)$, and define $\mathcal{G}_{1,0} : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$ as

$$\mathcal{G}_{1,0}(y_1) := \left\{ \xi_1 \in \mathbb{R}^{n_1} \,\middle|\, \exists\, \xi_2 \in \mathbb{R}^{n_2} \text{ satisfying } \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \mathcal{G} \begin{pmatrix} y_1 \\ 0 \end{pmatrix} \right\}.$$

Since $\operatorname{rint}(\operatorname{im} \widehat{\mathcal{M}}) \cap (\mathbb{R}^{n_1} \times \{0\}^{n_2}) \neq \emptyset$, there is a $\overline{y}_1 \in \mathbb{R}^{n_1}$ such that $\begin{pmatrix} \overline{y}_1 \\ 0 \end{pmatrix} \in \operatorname{rint}(\operatorname{im} \widehat{\mathcal{M}}) = \operatorname{rint}(\operatorname{dom} \mathcal{G})$, which shows that $\mathcal{G}_{1,0}$ is maximal monotone [13, Exercise 12.46].

For the given initial condition $\xi(0) \in \mathcal{S}_{\text{cons}}$, it follows that $\xi_1(0) \in \operatorname{im} \mathcal{G}_{1,0} = \operatorname{dom} \mathcal{G}_{1,0}^{-1}$. Hence, due to Theorem 1, there exists a unique Lipschitz continuous $\xi_1(t)$ that satisfies the differential inclusion

$$\dot{\xi}_1(t) \in -\mathcal{G}_{1,0}^{-1}(\xi_1(t)) \tag{8}$$

almost everywhere. It also follows from the same result that $\dot{\xi}_1 \in \mathcal{L}^\infty([0,\infty), \mathbb{R}^{n_1})$ and $\|\dot{\xi}_1\|_\infty \leq (\mathcal{G}_{1,0}^{-1})^0(\xi_1(0))$.

Next, for a fixed $\bar{y}_1 \in \mathbb{R}^{n_1}$, define $\mathcal{G}_{2,\bar{y}_1} : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ as

$$\mathcal{G}_{2,\bar{y}_1}(y_2) := \left\{ \xi_2 \in \mathbb{R}^{n_2} \,\middle|\, \exists\, \xi_1 \in \mathbb{R}^{n_1} \text{ satisfying } \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \mathcal{G} \begin{pmatrix} \bar{y}_1 \\ y_2 \end{pmatrix} \right\}.$$

Following the same arguments as in the case of $\mathcal{G}_{1,0}$, it can be shown that $\mathcal{G}_{2,\bar{y}_1}$ is maximal monotone. We define $\widehat{\mathcal{S}} : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ as

$$\widehat{\mathcal{S}}(t) := \mathcal{G}_{2,y_1(t)}(0),$$

where $y_1(t) := -\dot{\xi}_1(t)$ is the function satisfying (8). Clearly, $\widehat{\mathcal{S}}(t)$ is convex-valued as it is the image of a point under a maximal monotone operator. We choose $\xi_2(t) \in \widehat{\mathcal{S}}(t)$ as a part of the solution to (7).

To see that, the aforementioned construction of ξ_1 and ξ_2 is a solution to (7), it is seen that

$$\begin{pmatrix} \xi_1(t) \\ \xi_2(t) \end{pmatrix} \in \begin{pmatrix} \mathcal{G}_{1,0}(-\dot{\xi}_1(t)) \\ \mathcal{G}_{2,-\dot{\xi}_1(t)}(0) \end{pmatrix} \iff \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \end{pmatrix} \in \mathcal{G} \begin{pmatrix} -\dot{\xi}_1(t) \\ 0 \end{pmatrix}$$
$$\iff \begin{pmatrix} \dot{\xi}_1(t) \\ 0 \end{pmatrix} \in -\widehat{\mathcal{M}} \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \end{pmatrix}.$$

This completes the proof of Lemma 2. ■

We now combine Lemma 1 and Lemma 2 to write a formal proof of Theorem 2.

Proof of Theorem 2. Applying the transformation $\xi = Hx$ as described in Lemma 1, we obtain

$$\begin{pmatrix} \dot{\xi}_1 \\ 0 \end{pmatrix} \in -\widehat{\mathcal{M}}(\xi).$$

It is verified that the condition (6) leads to $\operatorname{rint}(\operatorname{im} \widehat{\mathcal{M}}) \cap (\mathbb{R}^{n_1} \times \{0\}^{n_2}) \neq \emptyset$, and that $\widehat{\mathcal{M}}^{-1}(\mathbb{R}^{n_1} \times \{0\}^{n_2}) = \mathcal{M}^{-1}(\operatorname{im} P)$. Hence, using Lemma 2, there exists a unique Lipschitz continuous ξ_1, and a time-varying convex set $\mathcal{S}_\xi : [0,\infty) \rightrightarrows \mathbb{R}^{n - \operatorname{rank} P}$ with $\xi_2(t) \in \mathcal{S}_\xi(t)$ as the admissible solution. It easily follows that

$$Px = H^\top \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ 0 \end{pmatrix}$$

showing that Px is Lipschitz continuous and the bound on $\|\frac{\mathrm{d}}{\mathrm{d}t} Px\|_\infty$ is obtained directly from the bound on $\|\dot{\xi}_1\|_\infty$. It is also observed that

$$|Px^1 - Px^2| \leq \left| H^\top(\xi_1^1 - \xi_1^2) \right| \leq \|H\| \, \|H^{-1}\| \, |x^1(0) - x^2(0)|.$$

For the multivalued component, one may choose $Q := \begin{bmatrix} 0 & I \end{bmatrix} H$, so that $Qx = \xi_2$, and the conclusion then follows from the statement of Lemma 2. ■

IV. Passive DAEs with Maximal Monotone Relations

We now address the problem of existence of solutions of DAEs of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} E\, x(t) = Ax(t) + Bz(t) \tag{9a}$$
$$w(t) = Cx(t) + Dz(t), \tag{9b}$$

where the variable $z(\cdot)$ is related to $w(\cdot)$ through the relation

$$w(t) \in \mathcal{M}(-z(t)) \tag{10}$$

and $\mathcal{M} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a maximal monotone operator.

Definition 3 (Solution concept): We call $(x, z, w) : [0,\infty) \to \mathbb{R}^{n+2m}$, a solution to system (9)-(10) if the following hold:

- Ex is absolutely continuous and the DAE (9a) holds for Lebesgue-almost every $t \geq 0$,
- $-z(t) \in \operatorname{dom}(\mathcal{M})$ for each $t \geq 0$,
- the relations (9b) and (10) holds for each $t \geq 0$.

The basic idea in studying the solutions of system (9)-(10) is to use the passivity notion to obtain an equivalent representation of the system in the inclusion of the form (1). We thus recall the passivity notion for standard DAEs and then describe the related transformations.

A. Passivity: Definition and Characterization

Definition 4: We call the system (9) passive with storage function $V : \mathbb{R}^n \to \mathbb{R}_+$ if

$$V(x(t_1)) \le V(x(t_0)) + \int_{t_0}^{t_1} z^\top(t) w(t)\, dt$$

for all t_0, t_1, $t_1 \ge t_0$ and (x, w, z) solves (9).

The following result is derived in [4], [7]:

Proposition 2: System (9) with minimal (E, A, B, C, D) is passive with storage function $x^\top K x$ for $K = K^\top \ge 0$ if, and only if, the following two statements hold:

- There exist matrices $S, T \in \mathbb{R}^{n\times n}$ that transform the system matrices in the special Weierstrass form:
$$(SET, SAT, SB, CT) = \left(\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} \right), \tag{11}$$
- The matrix K can be decomposed as
$$K = K^\top = \begin{bmatrix} K_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K_{33} \end{bmatrix} \ge 0 \tag{12}$$
and the submatrices in (11) satisfy the matrix inequaliy
$$\begin{bmatrix} A_1^\top K_{11} + K_{11} A_1 & K_{11} B_1 - C_1^\top \\ B_1^\top K_{11} - C & -(\widetilde{D} + \widetilde{D}^\top) \end{bmatrix} \le 0, \tag{13}$$
where $\widetilde{D} := D - C_2 B_2 - C_3 B_3$ and lastly
$$B_3^\top K_{33} = -C_2. \tag{14}$$

From the classical Kalman-Yakubovich-Popov lemma for ordinary differential equations, it readily follows from (13) that the quadruple $(A_1, B_1, C_1, \widetilde{D})$ describe a passive system.

B. Equivalence to DAIs

We now use the aforementioned passivity characterization to write the system (9)–(10) in the form of inclusion (1). The following lemma is crucial for that purpose.

Lemma 3: Let (E, A, B, C, D) be minimal and passive with quadratic storage function. There exists a full row rank matrix $P_{x_1} \in \mathbb{R}^{n_1\times n}$ where $n_1 < n$ and $\begin{bmatrix} A_1 & B_1 \\ C_1 & \tilde{D} \end{bmatrix} \in \mathbb{R}^{(n_1+m)\times(n_1+m)}$ with $(A_1, B_1, C_1, \tilde{D})$ passive such that for the DAE

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} I & 0 \\ 0 & -B_3^\top K_{33} B_3 \end{bmatrix} \begin{pmatrix} x_1 \\ z \end{pmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & \widetilde{D} \end{bmatrix} \begin{pmatrix} x_1 \\ z \end{pmatrix} - \begin{bmatrix} 0 \\ I \end{bmatrix} w \tag{15}$$

the following equivalence between (9) and (15) holds:

$$\{(P_{x_1} x, z, w) \,|\, (x, z, w) \text{ solves (9)}\} = \{(x_1, z, w) \,|\, (x_1, z, w) \text{ solves (15)} \wedge B_3 z \text{ is abs. cont.}\}.$$

Proof: From Proposition 2, we obtain the matrices S, T so that the solution triplet (x, z, w) of system (9) with $\mathrm{col}(x_1, x_2, x_3) = T^{-1} x$ satisfies

$$\dot{x}_1 = A_1 x_1 + B_1 z \tag{16a}$$
$$\dot{x}_3 = x_2 + B_2 z \tag{16b}$$
$$0 = x_3 + B_3 z \tag{16c}$$
$$w = C_1 x_1 + C_2 x_2 + C_3 x_3 + D z \tag{16d}$$

Next we use the notation introduced in Proposition 2, and recall that $C_2 = -B_3^\top K_{33}$ for some positive semidefinite K_{33}. Multiplying (16b) by $B_3^\top K_{33}$, and inserting (16c) as well as (16d), we arrive at

$$-\frac{\mathrm{d}}{\mathrm{d}t} B_3^\top K_{33} B_3 z = C_1 x_1 + (D - C_2 B_2 - C_3 B_3) z - w. \tag{17}$$

With P_{x_1} defined as

$$P_{x_1} := \begin{pmatrix} I & 0 & 0 \end{pmatrix} T^{-1} \tag{18}$$

the inclusion "$\subseteq$" is proven.

To show the converse inclusion let (x_1, z, w) be a solution of (15) and let $x_3 := -B_3 z$, $x_2 := -B_2 z - \frac{\mathrm{d}}{\mathrm{d}t} B_3 z$, then (x_1, x_2, x_3, z, w) solves (16) and hence $\left(T \begin{pmatrix} x_1^\top & x_2^\top & x_3^\top \end{pmatrix}^\top \quad z \quad w \right)$ solves (9). ■

This lemma is the key component in proving the following main result of this section:

Theorem 3: Assume that (E, A, B, C, D) is minimal and passive with quadratic storage function $x^\top K x$. Consider the DAI

$$\frac{\mathrm{d}}{\mathrm{d}t} \overline{P} \overline{x} \in -\overline{\mathcal{M}}(\overline{x}), \tag{19}$$

where using the notation from Lemma 3,

$$\overline{x} := \begin{pmatrix} x_1 \\ -z \end{pmatrix}, \quad \overline{P} := \begin{bmatrix} K_{11} & 0 \\ 0 & B_3^\top K_{33} B_3 \end{bmatrix}, \tag{20}$$

$$\overline{\mathcal{M}}(\overline{x}) = -\begin{bmatrix} K_{11} A_1 & -K_{11} B_1 \\ C_1 & -\widetilde{D} \end{bmatrix} \overline{x} + \begin{pmatrix} 0 \\ \mathcal{M}(-z) \end{pmatrix} \tag{21}$$

so that $\overline{P}$ is a positive semidefinite matrix and $\overline{\mathcal{M}}$ is maximal monotone. Then (9)–(10) is equivalent to (19) in the sense that

$$\{(P_{x_1} x, z) \,|\, (x, z, w) \text{ solves (9)}\} = \{(x_1, z) \,|\, (x_1, z) \text{ solves (19)} \wedge B_3 z \text{ is abs. cont.}\}$$

where P_{x_1} is defined in (18).

Proof: Once we have arrived at (15) in Lemma 3, then we can apply transformations (20) and (19) follows readily. By construction, $\overline{P}$ is positive semidefinite, and it only remains to show that $\overline{\mathcal{M}}$ is maximal monotone.

Since $(A_1, B_1, C_1, \widetilde{D})$ is passive, it follows that

$$\overline{A} = \begin{bmatrix} K_{11} A_1 & -K_{11} B_1 \\ C_1 & -\widetilde{D} \end{bmatrix}$$

is negative semidefinite. The result is immediate using the fact that

$$\begin{bmatrix} K_{11} A_1 + A_1^\top K_{11} & C_1^\top - K_{11} B_1 \\ C_1 - B_1^\top K_{11} & -(\widetilde{D} + \widetilde{D}^\top) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} K_{11} A_1 + A_1^\top K_{11} & K_{11} B_1 - C_1^\top \\ B_1^\top K_{11} - C_1 & -(\widetilde{D} + \widetilde{D}^\top) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

is negative semidefinite. The function $-\overline{A}\overline{x}$, being continuous, is thus maximal monotone with domain equal to entire space. Using [13, Chapter 12], the sum of two maximal monotone operators is maximal monotone, and its domain is the intersection of the corresponding domains. Thus, $\overline{\mathcal{M}}$ is maximal monotone with $\mathrm{dom}(\overline{\mathcal{M}}) = \mathrm{dom}(\mathcal{M})$. ■

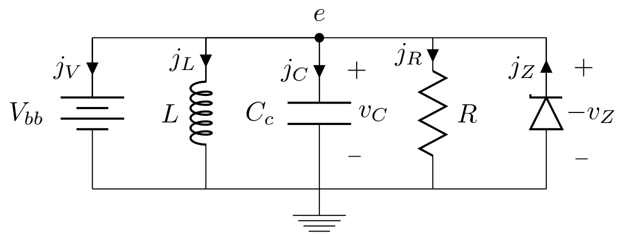


Fig. 1. Simple circuit example

V. Circuit Example

Let us consider the RLC circuit in Fig. 1 supplied by a constant voltage source (a battery) and interconnected with a Zener diode (see its input-output characteristic in Fig. 2). The modified nodal analysis approach [6] allows us to derive the following equations:

$$C_c \frac{\mathrm{d}e}{\mathrm{d}t} = -R^{-1}e - j_L - j_V + j_Z \tag{22a}$$

$$L\frac{\mathrm{d}j_L}{\mathrm{d}t} = e \tag{22b}$$

$$0 = e - V_{bb}, \tag{22c}$$

that can be rewritten as a DAE by assuming a state vector $\begin{pmatrix} e & j_L & j_V \end{pmatrix}$, an exogenous input vector $z = \begin{pmatrix} j_Z & V_{bb} \end{pmatrix}$ and $w = \begin{pmatrix} -v_Z & -j_V \end{pmatrix}$ as output of interest. Note that $w \in \mathcal{M}(-z)$ where $\mathcal{M}$ is a maximal monotone set-valued mapping whose graph is in $\mathbb{R}^2 \times \mathbb{R}^2$, see Figure 2a. The corresponding matrices are

$$E = \begin{pmatrix} C_c & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A = \begin{pmatrix} -R^{-1} & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{23a}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix},\ C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},\ D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{23b}$$

By choosing

$$S = \begin{pmatrix} 0 & L^{-1} & -L^{-1} \\ -1 & 0 & -R^{-1} \\ 0 & 0 & -C_c \end{pmatrix},\ T = \begin{pmatrix} 0 & 0 & -C_c^{-1} \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} \tag{24}$$

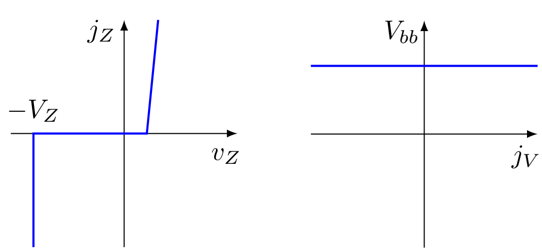
the representation (16) is obtained with $x_1 = j_L$, $x_2 = j_L + j_V$, $x_3 = -C_c e$:

$$\dot{x}_1 = L^{-1}V_{bb} \tag{25a}$$

$$\dot{x}_3 = x_2 - j_Z + R^{-1}V_{bb} \tag{25b}$$

$$0 = x_3 + C_c V_{bb}. \tag{25c}$$

In this case we have $A_1 = 0$, $B_1 = \begin{pmatrix} 0 & L^{-1} \end{pmatrix}$, $C_1^\top = \begin{pmatrix} 0 & 1 \end{pmatrix}$, $\tilde{D} = \begin{pmatrix} 0 & 1 \\ -1 & R^{-1} \end{pmatrix}$, $K_{11} = L$, $K_{22} = 0$, $K_{33} = C_c$.



(a) Zener diode voltage-current characteristic

(b) Battery current-voltage characteristic

Fig. 2. Set-valued mappings of the circuit example

The matrix inequality (13) is satisfied and the system evolves according to the differential inclusion (19) with

$$\overline{P} = \begin{pmatrix} L & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C_c \end{pmatrix}, \tag{26}$$

and

$$\overline{\mathcal{M}}(\overline{x}) = -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 1/R \end{pmatrix}\overline{x} + \begin{pmatrix} 0 \\ \mathcal{M}(-z) \end{pmatrix}. \tag{27}$$

VI. Conclusions

We have studied the existence of solutions of a novel class of dynamical systems described by differential-algebraic inclusions. Tools from the theory of maximal monotone operators have been used to prove existence of solutions globally in time. The results were used to study well-posedness of DAEs coupled with maximal monotone relations under certain passivity assumption. Applications of such systems are found in certain electrical circuits with nonsmooth devices such as diodes or transistors.

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