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ADDITION TO "THE QUASI-KRONECKER FORM FOR MATRIX PENCILS"*

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Abstract. We refine a result concerning singular matrix pencils and the Wong sequences. In our recent paper [T. Berger and S. Trenn, *SIAM J. Matrix Anal. Appl.*, 33 (2012), pp. 336–368] we have shown that the Wong sequences are sufficient to obtain a quasi-Kronecker form. However, we applied the Wong sequences again on the regular part to decouple the regular matrix pencil corresponding to the finite and infinite eigenvalues. The current paper is an addition to [T. Berger and S. Trenn, *SIAM J. Matrix Anal. Appl.*, 33 (2012), pp. 336–368], which shows that the decoupling of the regular part can be done already with the help of the Wong sequences of the original matrix pencil. Furthermore, we show that the complete Kronecker canonical form can be obtained with the help of the Wong sequences.

Key words. singular matrix pencil, Wong sequences, Kronecker canonical form, quasi-Kronecker form

AMS subject classifications. 15A22, 15A21

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1. Introduction.

In our recently published paper [2] we studied (singular) matrix pencils

$$sE - A \in \mathbb{K}^{m\times n}[s], \qquad \text{where } \mathbb{K} \text{ is } \mathbb{Q}, \mathbb{R}, \text{ or } \mathbb{C},$$

and showed how the *Wong sequences* [5]

$$\begin{aligned} \mathcal{V}_0 &:= \mathbb{K}^n, & \mathcal{V}_{i+1} &:= A^{-1}(E\mathcal{V}_i) \subseteq \mathbb{K}^n,\\ \mathcal{W}_0 &:= \{0\}, & \mathcal{W}_{i+1} &:= E^{-1}(A\mathcal{W}_i) \subseteq \mathbb{K}^n \end{aligned}$$

can be used to obtain a *quasi-Kronecker form*, where $M\mathcal{S} := \{\ Mx \in \mathbb{K}^m \mid x \in \mathcal{S}\ \}$ for some matrix $M \in \mathbb{K}^{m\times n}$ denotes the image of a subspace $\mathcal{S} \subseteq \mathbb{K}^n$ under M and $M^{-1}\mathcal{S} := \{\ x \in \mathbb{K}^n \mid Mx \in \mathcal{S}\ \}$ denotes the preimage of a subspace $\mathcal{S} \subseteq \mathbb{K}^m$ under M. The main feature of the quasi-Kronecker form is that it decouples the DAE $E\dot{x}(t) = Ax(t) + f(t)$ associated with the matrix pencil $sE - A$ into three parts: the underdetermined part, the regular part, and the overdetermined part. In particular, an explicit solution formula can be found just using the Wong sequences [2, Thm. 3.2]. However, for this result we applied the Wong sequences a second time (utilizing the results from [1]) to the regular part in order to decouple it further into the ODE part (slow dynamics, finite eigenvalues) and pure DAE part (fast dynamics, infinite eigenvalues). After the publication of [2] we became aware that this decoupling can in fact be done already with the Wong sequences of the original matrix pencil, and hence we are able to present a refined version of [2, Thms. 2.3 and 2.6]. Furthermore, the index of the regular part and the degrees of the infinite elementary divisors can

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be determined directly from the Wong sequences of the original matrix pencil (Proposition 2.4). We also show that the degrees of the finite elementary divisors can be derived using a modified version of the second Wong sequence (Proposition 2.6), and thus the complete Kronecker canonical form (KCF) can be obtained directly from these Wong sequences.

For a detailed literature review, notation, mathematical preliminaries, and further motivation we refer the reader to our main paper [2].

2. Main results.

THEOREM 2.1 (quasi-Kronecker triangular form (QKTF), refined version of [2, Thm. 2.3]). *Let $sE - A \in \mathbb{K}^{m\times n}[s]$, and consider the corresponding limits $\mathcal{V}^* := \bigcap_{i\in\mathbb{N}} \mathcal{V}_i$ and $\mathcal{W}^* := \bigcup_{i\in\mathbb{N}} \mathcal{W}_i$ of the Wong sequences. Choose any full rank matrices $P_1 \in \mathbb{K}^{n\times n_P}$, $R_1^J \in \mathbb{K}^{n\times n_J}$, $R_1^N \in \mathbb{K}^{n\times n_N}$, $Q_1 \in \mathbb{K}^{n\times n_Q}$, $P_2 \in \mathbb{K}^{m\times m_P}$, $R_2^J \in \mathbb{K}^{m\times m_J}$, $R_2^N \in \mathbb{K}^{m\times m_N}$, $Q_2 \in \mathbb{K}^{m\times m_Q}$ such that*

$$
\begin{aligned}
\operatorname{im} P_1 &= \mathcal{V}^* \cap \mathcal{W}^*, & \operatorname{im} P_2 &= E\mathcal{V}^* \cap A\mathcal{W}^*,\\
(\mathcal{V}^* \cap \mathcal{W}^*) \oplus \operatorname{im} R_1^J &= \mathcal{V}^*, & (E\mathcal{V}^* \cap A\mathcal{W}^*) \oplus \operatorname{im} R_2^J &= E\mathcal{V}^*,\\
\mathcal{V}^* \oplus \operatorname{im} R_1^N &= \mathcal{V}^* + \mathcal{W}^*, & E\mathcal{V}^* \oplus \operatorname{im} R_2^N &= E\mathcal{V}^* + A\mathcal{W}^*,\\
(\mathcal{V}^* + \mathcal{W}^*) \oplus \operatorname{im} Q_1 &= \mathbb{K}^n, & (E\mathcal{V}^* + A\mathcal{W}^*) \oplus \operatorname{im} Q_2 &= \mathbb{K}^m.
\end{aligned}
$$

Then it holds that $T_{\text{trian}} = [P_1, R_1^J, R_1^N, Q_1]$ and $S_{\text{trian}} = [P_2, R_2^J, R_2^N, Q_2]^{-1}$ are invertible and transform $sE - A$ into QKTF

$$
\begin{aligned}
(2.1)\quad &(S_{\text{trian}} E T_{\text{trian}}, S_{\text{trian}} A T_{\text{trian}})\\
&= \left(\begin{bmatrix} E_P & E_{PJ} & E_{PN} & E_{PQ} \\ 0 & E_J & E_{JN} & E_{JQ} \\ 0 & 0 & E_N & E_{NQ} \\ 0 & 0 & 0 & E_Q \end{bmatrix}, \begin{bmatrix} A_P & A_{PJ} & A_{PN} & A_{PQ} \\ 0 & A_J & A_{JN} & A_{JQ} \\ 0 & 0 & A_N & A_{NQ} \\ 0 & 0 & 0 & A_Q \end{bmatrix} \right),
\end{aligned}
$$

where the following hold:

(i) *$E_P, A_P \in \mathbb{K}^{m_P\times n_P}$, $m_P < n_P$, are such that* $\operatorname{rank}_{\mathbb{C}}(\lambda E_P - A_P) = m_P$ *for all $\lambda \in \mathbb{C} \cup \{\infty\}$;*

(ii) *$E_J, A_J \in \mathbb{K}^{m_J\times n_J}$, $m_J = n_J$, and* $\operatorname{rank}_{\mathbb{C}}(\lambda E_J - A_J) = n_J$ *for $\lambda = \infty$, i.e., E_J is invertible;*

(iii) *$E_N, A_N \in \mathbb{K}^{m_N\times n_N}$, $m_N = n_N$, and* $\operatorname{rank}_{\mathbb{C}}(\lambda E_N - A_N) = n_N$ *for all $\lambda \in \mathbb{C}$, i.e., A_N is invertible and $A_N^{-1}E_N$ is nilpotent;*

(iv) *$E_Q, A_Q \in \mathbb{K}^{m_Q\times n_Q}$, $m_Q > n_Q$, are such that* $\operatorname{rank}_{\mathbb{C}}(\lambda E_Q - A_Q) = n_Q$ *for all $\lambda \in \mathbb{C} \cup \{\infty\}$.*

Proof. Step 1: We show (2.1) and (i) and (iv).

As shown in [2, p. 340], we have the subspace inclusions $A\mathcal{V}^* \subseteq E\mathcal{V}^*$ and $E\mathcal{W}^* \subseteq A\mathcal{W}^*$, and from these it follows that

$$
\begin{aligned}
E(\mathcal{V}^* \cap \mathcal{W}^*) &\subseteq E\mathcal{V}^* \cap A\mathcal{W}^*, & A(\mathcal{V}^* \cap \mathcal{W}^*) &\subseteq E\mathcal{V}^* \cap A\mathcal{W}^*,\\
E\mathcal{V}^* &= E\mathcal{V}^*, & A\mathcal{V}^* &\subseteq E\mathcal{V}^*,\\
E(\mathcal{V}^* + \mathcal{W}^*) &\subseteq E\mathcal{V}^* + A\mathcal{W}^*, & A(\mathcal{V}^* + \mathcal{W}^*) &\subseteq E\mathcal{V}^* + A\mathcal{W}^*,\\
E\mathbb{K}^n &\subseteq \mathbb{K}^m, & A\mathbb{K}^n &\subseteq \mathbb{K}^m.
\end{aligned}
$$

These inclusions imply the solvability of

$$
\begin{aligned}
&EP_1 = P_2 E_P, && AP_1 = P_2 A_P,\\
&ER_1^J = P_2 E_{PJ} + R_2^J E_J, && AR_1^J = P_2 A_{PJ} + R_2^J A_J,\\
&ER_1^N = P_2 E_{PN} + R_2^J E_{JN} + R_2^N E_N, && AR_1^N = P_2 A_{PN} + R_2^J A_{JN} + R_2^N A_N,\\
&EQ_1 = P_2 E_{PQ} + R_2^J E_{JQ} + R_2^N E_{NQ} + Q_2 E_Q, && AQ_1 = P_2 A_{PQ} + R_2^J A_{JQ} + R_2^N A_{NQ} + Q_2 A_Q,
\end{aligned} \tag{2.2}
$$

which is equivalent to (2.1). The properties (i) and (iv) immediately follow from [2, Thm. 2.3], as the choice of bases here is more special.

Step 2: We show $(E\mathcal{V}^* \cap A\mathcal{W}^*) \oplus \operatorname{im} ER_1^J = E\mathcal{V}^*$.

As $\operatorname{im} R_1^J \subseteq \mathcal{V}^*$ it follows that $(E\mathcal{V}^* \cap A\mathcal{W}^*) + \operatorname{im} ER_1^J \subseteq E\mathcal{V}^*$. Invoking $E\mathcal{W}^* \subseteq A\mathcal{W}^*$, the opposite inclusion is immediate from

$$E\mathcal{V}^* = E((\mathcal{V}^* \cap \mathcal{W}^*) \oplus \operatorname{im} R_1^J) \subseteq E(\mathcal{V}^* \cap \mathcal{W}^*) + \operatorname{im} ER_1^J \subseteq (E\mathcal{V}^* + A\mathcal{W}^*) + \operatorname{im} ER_1^J.$$

It remains to show that the intersection is trivial. To this end let $x \in (E\mathcal{V}^* \cap A\mathcal{W}^*) \cap \operatorname{im} ER_1^J$, i.e., $x = Ey$ with $y \in \operatorname{im} R_1^J$. Further, $x \in E\mathcal{V}^* \cap A\mathcal{W}^* = E(\mathcal{V}^* \cap \mathcal{W}^*)$ (where the subspace equality follows from [2, Lem. 4.4]), and this yields that $x = Ez$ with $z \in \mathcal{V}^* \cap \mathcal{W}^*$, and thus $z - y \in \ker E \subseteq \mathcal{W}^*$. Hence, since $z \in \mathcal{W}^*$, it follows that $y \in \mathcal{W}^* \cap \operatorname{im} R_1^J = \{0\}$.

Step 3: We show $E\mathcal{V}^* \oplus \operatorname{im} AR_1^N = E\mathcal{V}^* + A\mathcal{W}^*$.

We immediately see that, since $A\mathcal{V}^* \subseteq E\mathcal{V}^*$,

$$
\begin{aligned}
E\mathcal{V}^* + A\mathcal{W}^* &= E\mathcal{V}^* + A\mathcal{V}^* + A\mathcal{W}^* = E\mathcal{V}^* + A(\mathcal{V}^* + \mathcal{W}^*)\\
&= E\mathcal{V}^* + A(\mathcal{V}^* + \operatorname{im} R_1^N) = E\mathcal{V}^* + A\mathcal{V}^* + A \operatorname{im} R_1^N = E\mathcal{V}^* + \operatorname{im} AR_1^N.
\end{aligned}
$$

In order to show that the intersection is trivial, let $x \in E\mathcal{V}^* \cap \operatorname{im} AR_1^N$, i.e., $x = Ay = Ez$ with $y \in \operatorname{im} R_1^N$ and $z \in \mathcal{V}^*$. Therefore, $y \in A^{-1}(E\mathcal{V}^*) = \mathcal{V}^*$ and $y \in \operatorname{im} R_1^N$, and thus $y = 0$.

Step 4: We show $m_J = n_J$ and $m_N = n_N$.

By Steps 2 and 3 we have that $m_J = \operatorname{rank} ER_1^J \le n_J$ and $m_N = \operatorname{rank} AR_1^N \le n_N$. In order to see that we have equality in both cases observe that $ER_1^J v = 0$ for some $v \in \mathbb{K}^{n_J}$ implies $R_1^J v \in \operatorname{im} R_1^J \cap \ker E = \{0\}$, since $\ker E \subseteq \mathcal{W}^*$, and hence $v = 0$ as R_1^J has full column rank; $AR_1^N v = 0$ for some $v \in \mathbb{K}^{n_N}$ implies $R_1^N v \in \operatorname{im} R_1^N \cap \ker A = \{0\}$, since $\ker A \subseteq \mathcal{V}^*$, and hence $v = 0$ as R_1^N has full column rank.

Step 5: We show that E_J and A_N are invertible.

For the first, assume that there exists $v \in \mathbb{K}^{n_J} \setminus \{0\}$ such that $E_J v = 0$. Then $ER_1^J v \overset{(2.2)}{=} P_2 E_{PJ} v$ and hence $ER_1^J v \in \operatorname{im} ER_1^J \cap \operatorname{im} P_2 \overset{\text{Step 2}}{=} \{0\}$, a contradiction with the fact that ER_1^J has full column rank (as shown in Step 4). In order to show that A_N is invertible, let $v \in \mathbb{K}^{n_N} \setminus \{0\}$ be such that $A_N v = 0$. Then $AR_1^N v \overset{(2.2)}{=} P_2 A_{PN} v + R_2^J A_{JN} v$ and hence $AR_1^N v \in \operatorname{im} AR_1^N \cap \operatorname{im}[P_2, R_2^J] \overset{\text{Step 3}}{=} \{0\}$, a contradiction with the fact that AR_1^N has full column rank (as shown in Step 4).

Step 6: It remains only to show that $A_N^{-1} E_N$ is nilpotent.

In order to prove this we will show that, for ℓ^* as in [2, (2.1)],

$$\forall\, i \in \{0, \dots, \ell^*\} : \ \mathcal{V}^* \oplus \operatorname{im} R_1^N (A_N^{-1} E_N)^i \subseteq \mathcal{V}^* + \mathcal{W}_{\ell^* - i}. \tag{2.3}$$

We show this by induction. For $i = 0$ the assertion is clear from the choice of R_1^N. Suppose (2.3) holds for some $i \in \{0, \ldots, \ell^* - 1\}$. Then

$$\begin{aligned}
&A(\mathcal{V}^* + \operatorname{im} R_1^N (A_N^{-1}E_N)^{i+1}) \subseteq A\mathcal{V}^* + \operatorname{im} AR_1^N (A_N^{-1}E_N)^{i+1} \\
&\overset{(2.2)}{\subseteq} E\mathcal{V}^* + \operatorname{im}(P_2 A_{PN} + R_2^J A_{JN} + R_2^N A_N)(A_N^{-1}E_N)^{i+1} \\
&\subseteq E\mathcal{V}^* + \underbrace{\operatorname{im} P_2 A_{PN}(A_N^{-1}E_N)^{i+1}}_{\subseteq E\mathcal{V}^*} + \underbrace{\operatorname{im} R_2^J A_{JN}(A_N^{-1}E_N)^{i+1}}_{\subseteq E\mathcal{V}^*} + \operatorname{im} R_2^N E_N (A_N^{-1}E_N)^i \\
&\overset{(2.2)}{\subseteq} E\mathcal{V}^* + \operatorname{im}(ER_1^N - P_2 E_{PN} - R_2^J E_{JN})(A_N^{-1}E_N)^i \\
&\subseteq E\mathcal{V}^* + \operatorname{im} ER_1^N (A_N^{-1}E_N)^i + \underbrace{\operatorname{im} P_2 E_{PN}(A_N^{-1}E_N)^i}_{\subseteq E\mathcal{V}^*} + \underbrace{\operatorname{im} R_2^J E_{JN}(A_N^{-1}E_N)^i}_{\subseteq E\mathcal{V}^*} \\
&\subseteq E(\mathcal{V}^* + \operatorname{im} R_1^N (A_N^{-1}E_N)^i) \overset{(2.3)}{\subseteq} E\mathcal{V}^* + E\mathcal{W}_{\ell^*-i} \subseteq E\mathcal{V}^* + A\mathcal{W}_{\ell^*-i-1}
\end{aligned}$$

and hence

$$\begin{aligned}
\mathcal{V}^* + \operatorname{im} R_1^N (A_N^{-1}E_N)^{i+1} &\subseteq A^{-1}(E\mathcal{V}^* + A\mathcal{W}_{\ell^*-i-1}) \\
&\subseteq A^{-1}(E\mathcal{V}^*) + \mathcal{W}_{\ell^*-i-1} \subseteq \mathcal{V}^* + \mathcal{W}_{\ell^*-i-1}.
\end{aligned}$$

Furthermore, we have

$$\mathcal{V}^* \cap \operatorname{im} R_1^N (A_N^{-1}E_N)^{i+1} \subseteq \mathcal{V}^* \cap \operatorname{im} R_1^N = \{0\},$$

and hence we have proved (2.3). Now (2.3) for $i = \ell^*$ yields $R_1^N (A_N^{-1}E_N)^{\ell^*} = 0$, and since R_1^N has full column rank, we may conclude that $(A_N^{-1}E_N)^{\ell^*} = 0$. ∎

Remark 2.2. In Theorem 2.1 the special choice of $R_2^J = ER_1^J$ and $R_2^N = AR_1^N$, which is feasible due to Steps 2 and 3 of the proof of Theorem 2.1, yields that (2.1) simplifies to

$$\left(\begin{bmatrix} E_P & 0 & E_{PN} & E_{PQ} \\ 0 & I_{n_J} & E_{JN} & E_{JQ} \\ 0 & 0 & N & E_{NQ} \\ 0 & 0 & 0 & E_Q \end{bmatrix}, \begin{bmatrix} A_P & A_{PJ} & 0 & A_{PQ} \\ 0 & A_J & 0 & A_{JQ} \\ 0 & 0 & I_{n_N} & A_{NQ} \\ 0 & 0 & 0 & A_Q \end{bmatrix} \right),$$

where N is nilpotent.

Corollary 2.3 (quasi-Kronecker form (QKF), refined version of [2, Thm. 2.6]). *Using the notation from Theorem* 2.1 *the following equations are solvable for matrices* $F_1, F_2, G_1, G_2, H_1, H_2, K_1, K_2, L_1, L_2, M_1, M_2$ *of appropriate size:*

$$\begin{aligned}
0 &= \begin{bmatrix} E_{JQ} \\ E_{NQ} \end{bmatrix} + \begin{bmatrix} E_J & E_{JN} \\ 0 & E_N \end{bmatrix} \begin{bmatrix} G_1 \\ F_1 \end{bmatrix} + \begin{bmatrix} G_2 \\ F_2 \end{bmatrix} E_Q, \\
0 &= \begin{bmatrix} A_{JQ} \\ A_{NQ} \end{bmatrix} + \begin{bmatrix} A_J & A_{JN} \\ 0 & A_N \end{bmatrix} \begin{bmatrix} G_1 \\ F_1 \end{bmatrix} + \begin{bmatrix} G_2 \\ F_2 \end{bmatrix} A_Q,
\end{aligned} \tag{2.4a}$$

$$\begin{aligned}
0 &= (E_{PQ} + E_{PN}F_1 + E_{PJ}G_1) + E_P K_1 + K_2 E_Q, \\
0 &= (A_{PQ} + A_{PN}F_1 + A_{PJ}G_1) + A_P K_1 + K_2 A_Q,
\end{aligned} \tag{2.4b}$$

$$\begin{aligned}
0 &= E_{JN} + E_J H_1 + H_2 E_N, \\
0 &= A_{JN} + A_J H_1 + H_2 A_N,
\end{aligned} \tag{2.4c}$$

$$\begin{aligned}
0 &= [E_{PJ}, E_{PN}] \begin{bmatrix} I & H_1 \\ 0 & I \end{bmatrix} + E_P[M_1, L_1] + [M_2, L_2] \begin{bmatrix} E_J & 0 \\ 0 & E_N \end{bmatrix}, \\
0 &= [A_{PJ}, A_{PN}] \begin{bmatrix} I & H_1 \\ 0 & I \end{bmatrix} + A_P[M_1, L_1] + [M_2, L_2] \begin{bmatrix} A_J & 0 \\ 0 & A_N \end{bmatrix},
\end{aligned} \tag{2.4d}$$

and for any such matrices let

$$S := \begin{bmatrix} I & -M_2 & -L_2 & -K_2 \\ 0 & I & -H_2 & -G_2 \\ 0 & 0 & I & -F_2 \\ 0 & 0 & 0 & I \end{bmatrix}^{-1} S_{\text{trian}}, \qquad T := T_{\text{trian}} \begin{bmatrix} I & M_1 & L_1 & K_1 \\ 0 & I & H_1 & G_1 \\ 0 & 0 & I & F_1 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

Then S and T are invertible and transform $sE - A$ into QKF

$$(SET, SAT) = \left(\begin{bmatrix} E_P & 0 & 0 & 0 \\ 0 & E_J & 0 & 0 \\ 0 & 0 & E_N & 0 \\ 0 & 0 & 0 & E_Q \end{bmatrix}, \begin{bmatrix} A_P & 0 & 0 & 0 \\ 0 & A_J & 0 & 0 \\ 0 & 0 & A_N & 0 \\ 0 & 0 & 0 & A_Q \end{bmatrix} \right), \tag{2.5}$$

where the block diagonal entries are the same as for the QKTF (2.1). In particular, the QKF (without the transformation matrices S and T) can be obtained with only the Wong sequences (i.e., without solving (2.4)). Furthermore, the QKF (2.5) is unique in the following sense:

$$(E, A) \cong (E', A') \;\Leftrightarrow\; \begin{aligned} &(E_P, A_P) \cong (E'_P, A'_P),\ (E_J, A_J) \cong (E'_J, A'_J), \\ &(E_N, A_N) \cong (E'_N, A'_N),\ (E_Q, A_Q) \cong (E'_Q, A'_Q), \end{aligned} \tag{2.6}$$

where $E'_P, A'_P, E'_J, A'_J, E'_N, A'_N, E'_P, A'_P$ are the corresponding blocks of the QKF of the matrix pencil $sE' - A'$.

Proof. We may choose $\lambda \in \mathbb{C}$ and M_λ of appropriate size such that $M_\lambda(A_N - \lambda E_N) = I$, and, due to [2, Lem. 4.14], for the solvability of (2.4c) it then suffices to consider the solvability of

$$E_J X A_N - A_J X E_N = -E_{JN} - (\lambda E_{JN} - A_{JN}) M_\lambda E_N,$$

which, however, is immediate from [2, Lem. 4.15]. The solvability of the other equations (2.4a), (2.4b), (2.4d) then follows as in the proof of Theorem 2.6 in [2].

Uniqueness in the sense of (2.6) can be established along lines similar to the proof of Theorem 2.6 in [2]. ∎

PROPOSITION 2.4 (index and infinite elementary divisors). *Consider the Wong sequences $\mathcal{V}_i$ and $\mathcal{W}_i$ and the notation from Theorem 2.1. Let*

$$\nu := \min \{\ i \in \mathbb{N} \mid \mathcal{V}^* + \mathcal{W}_i = \mathcal{V}^* + \mathcal{W}_{i+1}\ \}.$$

If $\nu \geq 1$, then ν is the index of nilpotency of $A_N^{-1}E_N$, i.e., $(A_N^{-1}E_N)^\nu = 0$ and $(A_N^{-1}E_N)^{\nu-1} \neq 0$. If $\nu = 0$, then $n_N = 0$; i.e., the pencil $sE_N - A_N$ is absent in the form (2.5).

Furthermore, if $\nu \geq 1$, let

$$\Delta_i := \dim(\mathcal{V}^* + \mathcal{W}_{i+1}) - \dim(\mathcal{V}^* + \mathcal{W}_i) \geq 0, \quad i = 0, 1, 2, \ldots \nu.$$

Then $\Delta_{i-1} \geq \Delta_i$ for $i = 1, 2, \ldots, \nu$, and, for $c = \Delta_0$, let the numbers $\sigma_1, \sigma_2, \ldots, \sigma_c \in \mathbb{N}$ be given by

$$\sigma_{c-\Delta_{i-1}+1} = \cdots = \sigma_{c-\Delta_i} = i, \quad i = 1, 2, \ldots, \nu,$$

where in case of $\Delta_{i-1} = \Delta_i$ the respective index range is empty.

Then $(E_N, A_N) \cong (N, I)$*, where* $N = \operatorname{diag}(N_{\sigma_1}, N_{\sigma_2}, \dots, N_{\sigma_c})$ *and, for* $\sigma \in \mathbb{N}$*,*

$$N_\sigma = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \in \mathbb{K}^{\sigma \times \sigma}.$$

Proof. As in the proof of [2, Thm. 2.9] we may assume, without loss of generality, that $sE - A$ is in KCF as in [2, Cor. 2.8]. Decomposing the Wong sequences into the four parts corresponding to each type of blocks, that is,

$$\mathcal{V}_i = \mathcal{V}_i^P \times \mathcal{V}_i^J \times \mathcal{V}_i^N \times \mathcal{V}_i^Q, \quad \mathcal{W}_i = \mathcal{W}_i^P \times \mathcal{W}_i^J \times \mathcal{W}_i^N \times \mathcal{W}_i^Q,$$

and supposing that $sE_P - A_P$, $sE_J - A_J = sI - J$, $sE_N - A_N = sN - I$, and $sE_Q - A_Q$ are in KCF, we find that the following hold:

(i) $\mathcal{V}_1^P = A_P^{-1}(\operatorname{im} E_P) = A_P^{-1}\mathbb{K}^{n_P} = \mathbb{K}^{n_P} \implies \mathcal{V}_i^P = \mathbb{K}^{n_P}$ for all $i \geq 0$.

(ii) $\mathcal{V}_1^J = J^{-1}\mathbb{K}^{n_J} = \mathbb{K}^{n_J} \implies \mathcal{V}_i^J = \mathbb{K}^{n_J}$ for all $i \geq 0$.

(iii) $\mathcal{V}_1^N = \operatorname{im} N$ and $\mathcal{V}_{i+1}^N = N\mathcal{V}_i^N \implies \mathcal{V}_i^N = \operatorname{im} N^i$ for all $i \geq 0$.

(iv) For the derivation of $\mathcal{V}_i^Q$, we assume for a moment that $sE_Q - A_Q$ consists only of one block, that is, $sE_Q - A_Q = \mathcal{Q}_\eta(s) = s\left[\begin{smallmatrix} 0 & \cdots & 0 \\ & I & \end{smallmatrix}\right] - \left[\begin{smallmatrix} & I & \\ 0 & \cdots & 0 \end{smallmatrix}\right]$ for some $\eta \in \mathbb{N}$. If $\eta = 0$, then by definition $\mathcal{V}_i^Q = \emptyset = \{0\}^0$ for all $i > 1$. Otherwise we have

$$\begin{aligned} \mathcal{V}_1^Q = A_Q^{-1}(\operatorname{im} E_Q) &= \left\{ x \in \mathbb{K}^\eta \;\middle|\; \exists\, y \in \mathbb{K}^\eta : \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix} \in \mathbb{K}^{\eta+1} \right\} \\ &= \{\, x \in \mathbb{K}^\eta \mid x_1 = 0 \,\}, \end{aligned}$$

and, iteratively, $\mathcal{V}_i^Q = \{\, x \in \mathbb{K}^\eta \mid x_1 = \dots = x_i = 0 \,\}$. In particular, $\mathcal{V}_\eta^Q = \{0\}^\eta$. For the general case, denote with $\eta_{\max} \in \mathbb{N}$ the maximal size of the $\mathcal{Q}_\eta(s)$ blocks in the KCF of $sE_Q - A_Q$. Then the above argument applied to each block in parallel yields $\mathcal{V}_{\eta_{\max}}^Q = \{0\}^{n_Q}$.

The above yields that

$$\mathcal{V}^* = \mathbb{K}^{n_P} \times \mathbb{K}^{n_J} \times \{0\}^{n_N} \times \{0\}^{n_Q}.$$

Now observe that the following hold:

(i) $\mathcal{W}_1^N = \ker N$ and $\mathcal{W}_{i+1}^N = N^{-1}(\mathcal{W}_i^N) \implies \mathcal{W}_i^N = \ker N^i$ for all $i \geq 0$.

(ii) $\mathcal{W}_1^Q = \ker E_Q = \{0\} \implies \mathcal{W}_i^Q = \{0\}^{n_Q}$ for all $i \geq 0$.

The assertion of the proposition is then immediate from

$$\mathcal{V}^* + \mathcal{W}_i = \mathbb{K}^{n_P} \times \mathbb{K}^{n_J} \times \ker N^i \times \{0\}^{n_Q}, \quad i \geq 0. \qquad \square$$

Remark 2.5. From Proposition 2.4 and [2, Thm. 2.9] we see that the degrees of the infinite elementary divisors and the row and column minimal indices (see, e.g., [3, 4] for these notions) corresponding to a matrix pencil $sE - A \in \mathbb{K}^{m \times n}[s]$ are fully determined by the Wong sequences corresponding to $sE - A$. It can also be seen from the representation of the Wong sequences for a matrix pencil in KCF that the degrees of the finite elementary divisors cannot be deduced from the Wong sequences. However, they can be derived from a modification of the second Wong sequence (similar to [1, Def. 3.3]), as shown in the following.

PROPOSITION 2.6 (finite elementary divisors). *Consider the Wong sequences* $\mathcal{V}_i$ *and* $\mathcal{W}_i$ *and the notation from Theorem* 2.1. *Denote with* $\sigma(sE_J - A_J) = \{\lambda_1, \lambda_2, \dots,$

$\lambda_k\} \subseteq \mathbb{C}$ the set of the $k \in \mathbb{N}$ distinct (generalized) eigenvalues of $sE_J - A_J$. Consider, for $\lambda \in \mathbb{C}$, the sequence

$$\mathcal{W}_0^\lambda := \{0\}, \quad \mathcal{W}_{i+1}^\lambda := (A - \lambda E)^{-1}(E\mathcal{W}_i^\lambda) \subseteq \mathbb{K}^n. \tag{2.7}$$

Then we have, for all $\lambda \in \mathbb{C}$, the characterization

$$\lambda \notin \sigma(sE_J - A_J) \iff \mathcal{W}_1^\lambda \subseteq \mathcal{W}^*. \tag{2.8}$$

Consider now the notation from [2, Cor. 2.8], and reorder $\mathcal{J}_{\rho_1}(s), \ldots, \mathcal{J}_{\rho_b}(s)$ as $\mathcal{J}_{\rho_{1,1}}^{\lambda_1}(s), \ldots, \mathcal{J}_{\rho_{b_1,1}}^{\lambda_1}(s), \mathcal{J}_{\rho_{1,2}}^{\lambda_2}(s), \ldots, \mathcal{J}_{\rho_{b_2,2}}^{\lambda_2}(s), \ldots, \mathcal{J}_{\rho_{1,k}}^{\lambda_k}(s), \ldots, \mathcal{J}_{\rho_{b_k,k}}^{\lambda_k}(s)$ with $\rho_{1,j} \leq \cdots \leq \rho_{b_j,j}$ for all $j = 1, \ldots, k$, where

$$\mathcal{J}_{\rho_{i,j}}^{\lambda_j}(s) = sI - \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix} \in \mathbb{C}^{\rho_{i,j} \times \rho_{i,j}}[s], \quad j = 1 \ldots, k, \ i = 1, \ldots, b_j.$$

Let

$$\Delta_i^j := \dim(\mathcal{W}^* + \mathcal{W}_{i+1}^{\lambda_j}) - \dim(\mathcal{W}^* + \mathcal{W}_i^{\lambda_j}), \quad j = 1, \ldots, k, \quad i = 0, 1, 2, \ldots .$$

Then $\Delta_0^j = b_j$, $\Delta_{i-1}^j \geq \Delta_i^j$ and

$$\rho_{b_j - \Delta_{i-1}^j + 1, j} = \cdots = \rho_{b_j - \Delta_i^j, j} = i, \quad j = 1, \ldots, k, \quad i = 1, 2, 3, \ldots .$$

Proof. Similar to the proof of Proposition 2.4 we may consider $sE - A$ in KCF. Then

$$\mathcal{W}^* = \mathbb{K}^{n_P} \times \{0\} \times \mathbb{K}^{n_N} \times \{0\}.$$

The proof now follows from the observation that, for all $\lambda \in \mathbb{C}$ and $i \in \mathbb{N}$,

$$\mathcal{W}^* + \mathcal{W}_i^\lambda = \mathbb{K}^{n_P} \times \left(\bigtimes_{\substack{j=1,\ldots,k \\ l=1,\ldots,b_k}} \left(\ker \mathcal{J}_{\rho_{l,j}}^{\lambda_j}(\lambda) \right)^i \right) \times \mathbb{K}^{n_N} \times \{0\}^{n_Q}$$

and $\ker \mathcal{J}_{\rho_{l,j}}^{\lambda_j}(\lambda) = \{0\}$ for $\lambda \neq \lambda_j$. ∎

Remark 2.7 (Jordan canonical form). In a case of a pencil $sI - A$, the following simplifications can be made in Proposition 2.6: $\mathcal{W}^* = \{0\}$, and hence $\mathcal{W}_i^\lambda = \ker(A - \lambda I)^i$. Then (2.8) becomes the classical eigenvalue definition

$$\lambda \text{ is an eigenvalue of } A \iff \ker(A - \lambda I) \neq \{0\}.$$

Furthermore,

$$\Delta_i^j = \dim\ker(A - \lambda_j I)^{i+1} - \dim\ker(A - \lambda_j I)^i,$$

which is the well-known formula for the number of Jordan blocks of size $i + 1$ or greater corresponding to the eigenvalue λ_j of A.

Remark 2.8 (determination of the KCF). The results presented so far show that the KCF of a pencil $sE - A$ (without the corresponding transformation matrices) is completely determined by the Wong sequences:

(i) The row and column minimal indices η_i and ε_i are given by [2, Thm. 2.9], which directly give the KCF of the singular part of the matrix pencil.

(ii) The degrees σ_i of the infinite elementary divisors are given by Proposition 2.4 yielding the KCF of the matrix pencil $sE_N - A_N$.

(iii) Finally, the finite eigenvalues can be determined by deriving the roots of $\det(\lambda E_J - A_J)$ or using (2.8), and the degrees ρ_i of the finite elementary divisors (corresponding to the above eigenvalues) are given by Proposition 2.6. This yields the Jordan canonical form of $E_J^{-1} A_J$ completing the KCF.

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