Edge-wise funnel synchronization

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Problem statement

Given

- N agents with individual scalar dynamics:

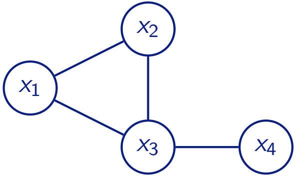
$$\dot{x}_i = f_i(t, x_i) + u_i$$

- undirected connected coupling-graph $G = (V, E)$
- local feedback

Desired

Control design for practical synchronization

$$x_1 \approx x_2 \approx \ldots \approx x_n$$



$$u_1 = F_1(x_1, x_2, x_3)$$
$$u_2 = F_2(x_2, x_1, x_3)$$
$$u_3 = F_3(x_3, x_1, x_2)$$
$$u_4 = F_4(x_4, x_3)$$

A „high-gain“ result

Let $\mathcal{N}_i := \{ j \in V \mid (j,i) \in E \}$ and $d_i := |\mathcal{N}_i|$ and $\mathcal{L}$ be the Laplacian of G.

Diffusive coupling

$$u_i = -k \sum_{j\in\mathcal{N}_i} (x_i - x_j) \quad \text{or, equivalently,} \quad u = -k\ \mathcal{L}\ x$$

Theorem (Practical synchronization, Kim et al. 2013)

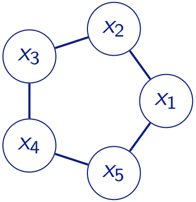
Assumptions: G connected, all solutions of average dynamics

$$\dot{s}(t) = \frac{1}{N}\sum_{i=1}^{N} f_i(t, s(t))$$

remain bounded. Then $\forall \varepsilon > 0\ \exists K > 0\ \forall k \geq K$: Diffusive coupling results in

$$\boxed{\limsup_{t\to\infty} |x_i(t) - x_j(t)| < \varepsilon \quad \forall i,j \in V}$$

Example (taken from Kim et al. 2015)



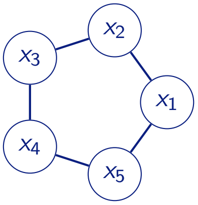
Simulations in the following for $N = 5$ agents with dynamics

$$f_i(t, x_i) = (-1 + \delta_i)x_i + 10 \sin t + 10m_i^1 \sin(0.1t + \theta_i^1) + 10m_i^2 \sin(10t + \theta_i^2),$$

with randomly chosen parameters $\delta_i, m_i^1, m_i^1 \in \mathbb{R}$ and $\theta_i^1, \theta_i^2 \in [0, 2\pi]$.

Parameters identical in all following simulations, in particular $\delta_2 > 1$, hence agent 2 has unstable dynamics (without coupling).

Example (taken from Kim et al. 2015)

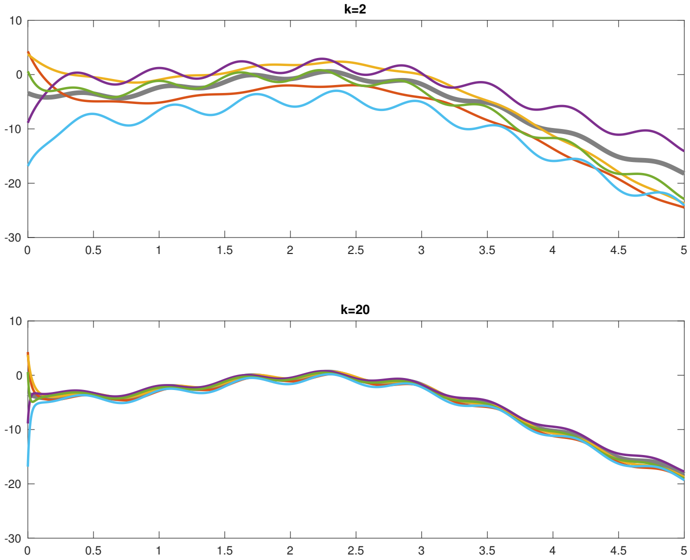


$$u = -k\,\mathcal{L}\,x$$

gray curve:

$$\dot{s}(t) = \frac{1}{N}\sum_{i=1}^{N} f_i(t, s(t))$$

$$s(0) = \frac{1}{N}\sum_{i=1}^{N} x_i(0)$$



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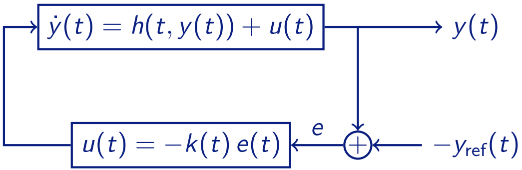
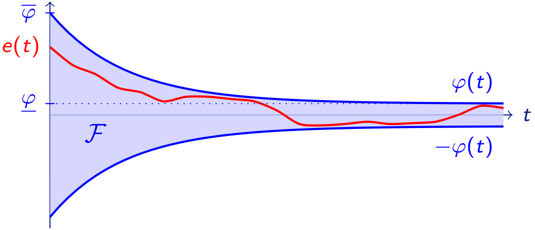
Reminder Funnel Controller





Theorem (Practical tracking, Ilchmann et al. 2002)

Funnel Control

$$k(t) = \frac{1}{\varphi(t) - |e(t)|}$$

works, in particular, errors remains within funnel for all times.

Basic idea for funnel synchronization

$$u = -k\, \mathcal{L}\, x \quad \longrightarrow \quad u = -k(t)\, \mathcal{L}\, x$$

Content

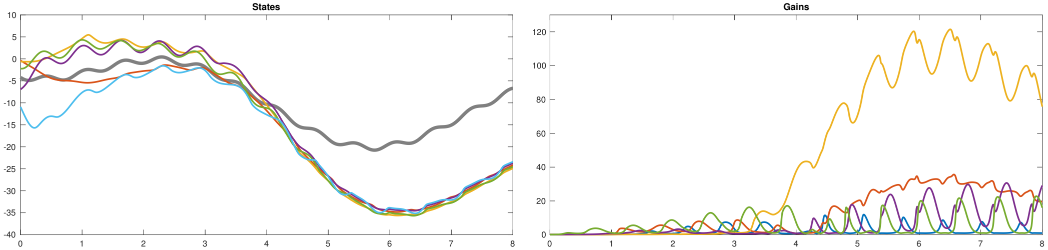
Approach from Shim & T. 2015

Local error

$$u_i = -k \sum_{j:(i,j)\in E} x_i - x_j = -k \left(d_i x_i - \sum_{j:(i,j)\in E} x_j \right) =: -k\, d_i\, (x_i - \overline{x}_i) =: -k_i\ e_i$$

Funnel synchronization feedback rule

$$u_i(t) = -k_i(t) e_i(t) \quad \text{with} \quad k_i(t) = \frac{1}{\varphi(t) - |e_i(t)|}$$



Unpredictable limit trajectory

Problems

Synchronization occurs as desired, but

- No proof available yet
- Non-predictable limit trajectory

Laplacian feedback

Diffusive coupling

$$u = -k\,\mathcal{L}\,x$$

has Laplacian feedback matrix $k\mathcal{L}$

Non-Laplacian feedback

Funnel synchronization

$$u = -K(t)\,\mathcal{L}\,x = -\begin{bmatrix} k_1(t) & & & \\ & k_2(t) & & \\ & & \ddots & \\ & & & k_N(t) \end{bmatrix}\mathcal{L}\,x$$

has non-Laplacian feedback matrix $K(t)\mathcal{L}$, in particular $[1, 1, \dots, 1]^\top$ is not a left-eigenvector of $K(t)\mathcal{L}$.

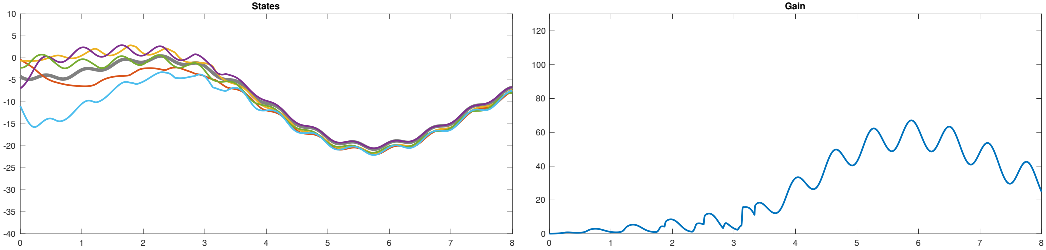
Weakly centralized Funnel synchronization, Shim & T. 2015

Restoring Laplacian feedback structure

Weakly decentralized funnel synchronization

$$u = -k_{\max}(t)\mathcal{L}x \quad \text{with} \quad k_{\max}(t) := \max_i k_i(t)$$

again has (time-varying) Laplacian feedback matrix $-k_{\max}(t)\mathcal{L}$.



Problem

Each agent needs knowledge of gains of all other agents!

Content

Diffusive coupling revisited

Diffusive coupling for weighted graph

$$u_i = -k \sum_{i}^{N} \alpha_{ij} \cdot (x_i - x_j) \quad \longrightarrow \quad u_i = -\sum_{i}^{N} k_{ij} \cdot \alpha_{ij} \cdot (x_i - x_j)$$

where $\alpha_{ij} = \alpha_{ji} \in \{0, 1\}$ is the weight of edge (i, j)

Conjecture

If $k_{ij} = k_{ji}$ are all sufficiently large, then practical synchronization occurs with predictable limit trajectory s.

Proof technique from Kim et al. 2013 should still work in this setup.

Adjusted proof technique of Kim et al. 2013

Consider coordinate transformation $\begin{pmatrix} \xi \\ r \end{pmatrix} = \frac{1}{N} \begin{bmatrix} 1_N^\top \\ R(k_{ij}) \end{bmatrix} x$, then closed loop has the form

$$\begin{aligned} \dot{\xi} &= \frac{1}{N} 1_N^\top f\left(t, 1_N \xi + Qr\right) \\ \dot{r} &= -\Lambda(k_{ij})\, r + R(k_{ij}) f\left(t, 1_N \xi + Qr\right) \end{aligned}$$

Show that $r \to 0$, then $\xi \to s$ where

$$\dot{s} = \frac{1}{N} 1_N^\top f\left(t, 1_N s\right)$$

Problem

Coordinate transformation depends on k_{ij}
$\rightarrow$ Approach breaks down when k_{ij} becomes time/state-dependent

Edgewise Funnel synchronization

Diffusive coupling → edgewise Funnel synchronization

$$u_i = -\sum_{i}^{N} k_{ij} \cdot \alpha_{ij} \cdot (x_i - x_j) \quad \longrightarrow \quad u_i = -\sum_{i}^{N} k_{ij}(t) \cdot \alpha_{ij} \cdot (x_i - x_j)$$

Edgewise error feedback

$$k_{ij}(t) = \frac{1}{\varphi(t) - |e_{ij}|}, \quad \text{with} \quad e_{ij} := x_i - x_j$$

Properties:

- Decentralized, i.e. u_i only depends on state of neighbors
- Symmetry, $k_{ij} = k_{ji}$
- Laplacian feedback, $u = -\mathcal{L}_K(t,x)x$

No finite escape time

Assumption 1

For $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, assume that

$$\dot{\alpha} = \max_{\|z\|_2 = \sqrt{2\alpha}} z^\top f(t,z), \qquad \alpha(0) \geq 0,$$

has no finite escape time.

Lemma (Shim & Trenn 2015, CDC)

Any nonlinear system

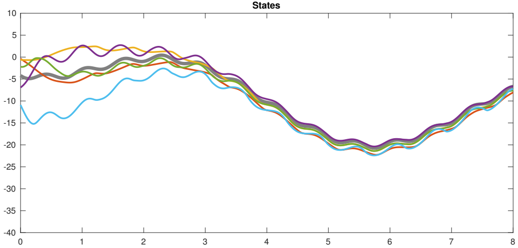
$$\dot{x} = f(t,x) - M(t,x)x$$

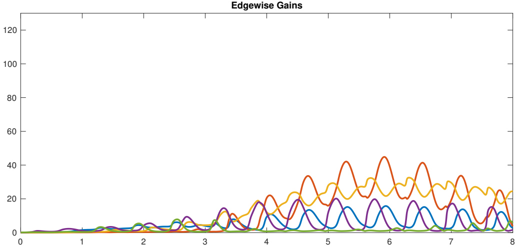
with positive semi-definite $M(t,x)$ where f satisfying Assumption 1 has no finite-escape time (in x).

Corollary

Under Assumption 1, edgewise funnel control has no finite escape time (in x).

Simulation and Discussion





Discussion

- Synchronization occurs
- Predictable limit trajectory (global consensus)
- Local feedback law
- No proofs available yet
- Restricted to scalar systems so far
- Restricted to undirected graphs so far