Passive DAEs and maximal monotone operators

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joint work with
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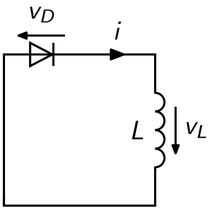
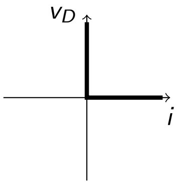
Motivation: Electrical circuits with ideal diodes



$$\tfrac{\mathrm{d}}{\mathrm{d}t} i = L v_D$$
$$0 \leq i \perp v_D \geq 0$$



Linear complementarity systems

$$\dot{x} = Ax + Bz$$
$$w = Cx + Dz$$
$$0 \leq z \perp w \geq 0$$

Theorem (Camlibel et al. 1999)

(A, B, C, D) *passive*
$\Downarrow$
Existence & uniqueness of solutions

Reformulation: $0 \leq i \perp v_D \geq 0 \quad \Leftrightarrow \quad v_D \in \begin{cases} \emptyset, & i < 0, \\ [0,\infty), & i = 0, \\ \{0\}, & i > 0. \end{cases}$

Set-valued constraints

$$\dot{x} = Ax + Bz$$
$$w = Cx + Dz$$
$$w \in \mathcal{F}(-z)$$

Theorem (Camlibel et al. 2015)

(A,B,C,D) *passive and*
$\mathcal{F}$ *maximal-monotone*
$\Downarrow$
Existence & Uniqueness of solutions

Question

Generalization to DAEs

$$\begin{aligned} E\dot{x} &= Ax + Bz \\ w &= Cx + Dz \\ w &\in \mathcal{F}(-z) \end{aligned}$$

(E, A, B, C, D) passive & $\mathcal{F}$ maximal-monotone $\overset{?}{\Rightarrow}$ existence & uniqueness of solutions

Maximal-monotone operators

Definition (Monotonicity)

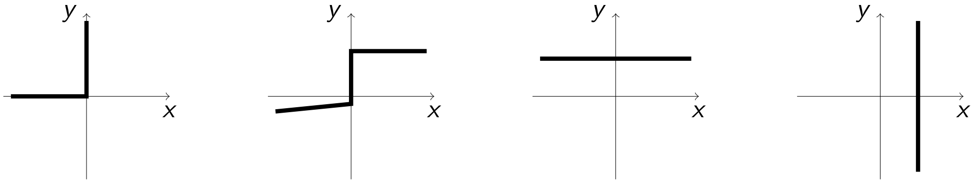
$\mathcal{M} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is monotone $:\Leftrightarrow$

$$\forall y_1 \in \mathcal{M}(x_1),\ y_2 \in \mathcal{M}(x_2) : \quad \langle y_2 - y_1, x_2 - x_1 \rangle \geq 0$$

A monotone $\mathcal{M} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal $:\Leftrightarrow$

$$\forall \widetilde{\mathcal{M}} \supset \mathcal{M} : \quad \widetilde{\mathcal{M}} \text{ is not monotone}$$

Examples for scalar maximal-monotone operators:



Maximal-monotone operators

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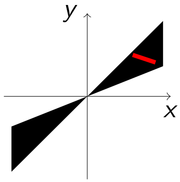
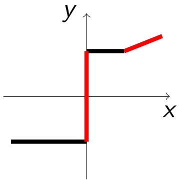
Non-monotone example:



Non-maximal example:



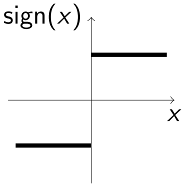
Maximal-monotone operators and differential inclusions

Theorem (Brezis 1973)

$\mathcal{M} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *max.-monotone* $\Rightarrow$ $\dot{x} \in -\mathcal{M}(x)$, $x(0) = x_0 \in \mathrm{dom}(\mathcal{M})$, *is uniquely solvable*

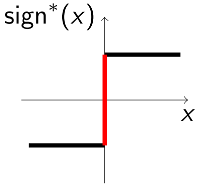
No global solution:

$$\dot{x} = -\operatorname{sign}(x) := \begin{cases} -1, & x \geq 0, \\ 1, & x < 0 \end{cases}$$



Global solutions (Philipov-solutions)

$$\dot{x} \in -\operatorname{sign}^*(x) := \begin{cases} -1, & x > 0, \\ [-1,1], & x = 0, \\ 1, & x < 0 \end{cases}$$



Linear systems with set-valued constraints

We have:

$$\begin{aligned} \dot{x} &= Ax + Bz \\ w &= Cx + Dz \\ w &\in \mathcal{F}(-z) \end{aligned} \qquad \Longleftrightarrow \qquad \dot{x} \in -\mathcal{M}(x)$$

where

$$\mathcal{M}(x) := -Ax + B(\mathcal{F} + D)^{-1}(Cx).$$

Passivity and maximal-monotonicity

(A, B, C, D) passive & $\mathcal{F}$ maximal-monotone $\quad\Rightarrow\quad$ $\mathcal{M}$ is maximal-monotone

Linear systems with set-valued constraints

We have:

$$\begin{aligned} E\dot{x} &= Ax + Bz \\ w &= Cx + Dz \\ w &\in \mathcal{F}(-z) \end{aligned} \qquad \Longleftrightarrow \qquad \dot{x} \in -E^{-1}\mathcal{M}(x)$$

where

$$\mathcal{M}(x) := -Ax + B(\mathcal{F}+D)^{-1}(Cx).$$

Maximal-monotonicity is lost

(E, A, B, C, D) passive & $\mathcal{F}$ maximal-monotone $\not\Rightarrow$ $E^{-1}\mathcal{M}(x)$ is maximal-monotone

$$\begin{aligned} \dot{x}_1 &= z \\ \dot{x}_3 &= x_2 + z \\ 0 &= x_3 + z \\ w &= x_1 \\ -z &\in \mathcal{F}^{-1}(w) := \max\{0, w\} \end{aligned} \qquad \Longleftrightarrow \qquad \dot{x} \in -\begin{pmatrix} x_3 \\ \mathbb{R} \\ -x_2 + x_3 \end{pmatrix}, \text{ for } x_3 = \max\{0, x_1\}, \ \emptyset \text{ otherwise}$$

not monotone, consider e.g. $\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \in E^{-1}\mathcal{M}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)$

Passivity: Definitions and important consequences

Definition (Passivity)

$$\begin{aligned} E\dot{x} &= Ax + Bz \\ w &= Cx + Dz \end{aligned} \quad \text{passive} \quad :\Leftrightarrow \quad \exists V : \mathbb{R}^n \to \mathbb{R}_+ : \quad V(x(t_1)) \leq V(x(t_0)) + \int_{t_0}^{t_1} z^\top w$$

Lemma (Passivity & special quasi-Weierstrass-form, Freund & Jarre 2004)

(E, A, B, C, D) passive (and minimal) $\Rightarrow$ *$\exists S, T$ invertible:*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \right)$$

In particular, a (minimal) passive DAE is either an ODE or an index-2-DAE.

Passivity: Charakterisation

Theorem (Passivity & LMIs, Camlibel & Frasca 2009)

$$(E,A,B,C,D) = \left(\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix}, D \right)$$ *is passive with* $V(x) = x^\top K x \Leftrightarrow$

1. $K = K^\top = \begin{bmatrix} K_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K_{33} \end{bmatrix} \geq 0$
2. $(A_1, B_1, C_1, D - C_2 B_2 - C_3 B_3)$ *is passive, i.e. the following LMI holds:*
$$\begin{bmatrix} A_1^\top K_{11} + K_{11} A_1 & K_{11} B_1 - C_1^\top \\ B_1^\top K_{11} - C_1 & -(\widetilde{D} + \widetilde{D}^\top) \end{bmatrix} \leq 0, \quad \textit{where } \widetilde{D} = D - C_2 B_2 - C_3 B_3$$
3. $B_3^\top K_{33} = -C_2$

Elimination of variables

Theorem (Camlibel, Iannelli, Tanwani, T. 2016)

(x, z, w) *solves passive and minimal* (E, A, B, C, D) *with constraint* $w \in \mathcal{F}(-z)$ $\iff$

$$\overline{x} := \begin{pmatrix} x_1 \\ -z \end{pmatrix} \quad \textit{solves} \quad P\dot{\overline{x}} \in -\overline{\mathcal{M}}(\overline{x}),$$

where

$$P := \begin{bmatrix} K_{11} & 0 \\ 0 & B_3^\top K_{33} B_3 \end{bmatrix} \text{ symmetric and positive-semidefinit}$$

and

$$\overline{\mathcal{M}}(\overline{x}) := -\begin{bmatrix} K_{11}A_1 & -K_{11}B_1 \\ C_1 & -\widetilde{D} \end{bmatrix}\overline{x} + \begin{pmatrix} 0 \\ \mathcal{F}(-z) \end{pmatrix} \text{ maximal-monotone}$$

New class of differential-inclusions

Differential-algebraic-inclusions (DAIs)

$$P\dot{x} \in -\mathcal{M}(x) \qquad \textbf{(DAI)}$$

Theorem (Camlibel, Iannelli, Tanwani, T. 2016)

*Consider (**DAI**) with $P \geq 0$ and max.-mon. $\mathcal{M}$. Then:*

1. *For every initial condition $x(0) = x_0$ with $x_0 \in \mathcal{M}^{-1}(\operatorname{im} P)$ a global solution $x : [0, \infty) \to \mathbb{R}^n$ with absolute-continuos Px exists*
2. *Stability in the following sense holds:*

$$\left\| Px^1(t) - Px^2(t) \right\| \leq c\|x^1(0) - x^2(0)\|,$$

in particular, Px uniquely determined by initial value.

Summary

$$\boxed{\begin{aligned}\dot{x} &= Ax + Bz\\ w &= Cx + Dz\\ w &\in \mathcal{F}(-z)\end{aligned}} \qquad \boxed{\begin{aligned}E\dot{x} &= Ax + Bz\\ w &= Cx + Dz\\ w &\in \mathcal{F}(-z)\end{aligned}}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\boxed{\dot{x} \in -\mathcal{M}(x)} \qquad \boxed{P\dot{\overline{x}} \in -\overline{\mathcal{M}}(\overline{x})}$$

- Passivity preserves maximal-monotonicity
- DAEs lead to maximal-monotene differential-algebraic-inclusion
- Uniqueness of solutions is lost, but global existence is guaranteed
- Further questions:
  - External inputs (works for ODE case)
  - How important is positive-semi-definitness and symmetry of P?
  - Physical interpretation of non-uniqueness?