Funnel synchronization for multi agent systems

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Technische Universität Kaiserslautern

Inhalt

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2. High-gain and funnel control
3. Simulations
4. Weakly centralized Funnel synchronization

Problem statement

Given

- N Agents with individual scalar dynamics:

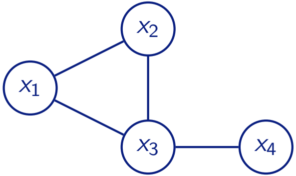
$$\dot{x}_i = f_i(t, x_i) + u_i$$

- undirected connected coupling-graph $G = (V, E)$
- agents know average of neighbor states

Desired

Control design for practical synchronization

$$x_1 \approx x_2 \approx \ldots \approx x_n$$



$$\bar{x}_1 := \frac{1}{2}(x_2 + x_3)$$

$$\bar{x}_2 := \frac{1}{2}(x_1 + x_3)$$

$$\bar{x}_3 := \frac{1}{3}(x_1 + x_2 + x_4)$$

$$\bar{x}_4 := x_3$$

A „high-gain“ result

Let $\mathcal{N}_i := \{\, j \in V \mid (j,i) \in E \,\}$ and $d_i := |\mathcal{N}_i|$.

Diffusive coupling

$$u_i = -k \sum_{j \in \mathcal{N}_i} (x_i - x_j) = -k d_i (x_i - \overline{x}_i)$$

Theorem (Practical synchronization, Kim et al. 2013)

Assumptions: G connected, $(t,a) \mapsto f_i(t,a)$ bounded in t and global Lipschitz in a, all solutions of average dynamics

$$\dot{s}(t) = \frac{1}{N} \sum_{i=1}^{N} f_i(t, s(t))$$

remain bounded. Then $\forall \varepsilon > 0\ \exists K > 0\ \forall k \geq K$: Diffusive coupling results in

$$\boxed{\limsup_{t \to \infty} |x_i(t) - x_j(t)| < \varepsilon \quad \forall i,j \in V}$$

Remarks on high-gain result

Common trajectory

It even holds that

$$\limsup_{t\to\infty} |x_i(t) - s(t)| < \varepsilon/2,$$

where $s(\cdot)$ is the solution of

$$\dot{s}(t) = \frac{1}{N}\sum_{i=1}^{N} f_i(t, s(t)), \qquad s(0) = \frac{1}{N}\sum_{i=1}^{N} x_i.$$

Independent of coupling structure and amplification k.

Error feedback

With $e_i := x_i - \overline{x}_i$ diffusive coupling has the form

$$u_i = -k_i e_i$$

Attention: $e_i \neq x_i - s$, in particular, agents do not know „limit trajectory“ $t \mapsto s(t)$

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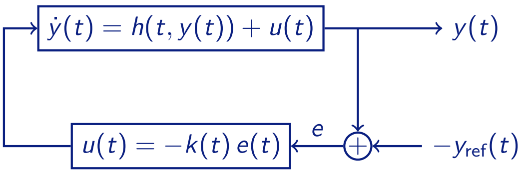
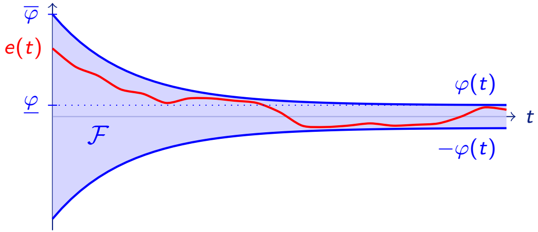
Reminder Funnel Controller





Theorem (Practical tracking, Ilchmann et al. 2002)

Funnel Control

$$k(t) = \frac{1}{\varphi(t) - |e(t)|}$$

works.

Funnel synchronization

Reminder diffusive coupling: $u_i = -k_i e_i$ with $e_i = x_i - \overline{x}_i$.

Combine diffusive coupling with Funnel Controller

$$u_i(t) = -k_i(t)\, e_i(t) \quad \text{mit} \quad k_i(t) = \frac{1}{\varphi(t) - |e_i(t)|}$$

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Example

Simulations in the following for $N = 5$ agents with dynamics

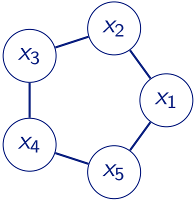
$$f_i(t, x_i) = (-1 + \delta_i)x_i + 10\sin t + 10m_i^1 \sin(0.1t + \theta_i^1) + 10m_i^2 \sin(10t + \theta_i^2),$$

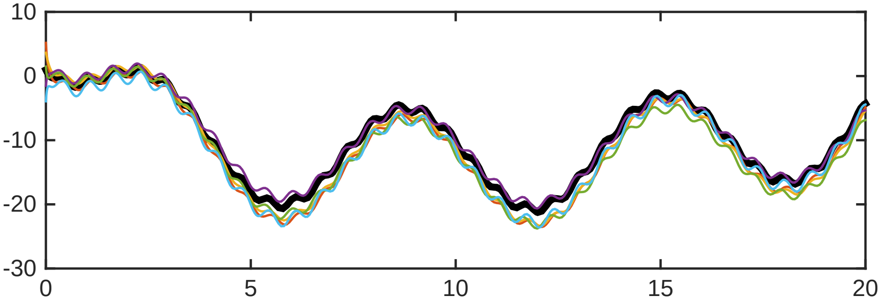
with randomly chosen parameters $\delta_i, m_i^1, m_i^1 \in \mathbb{R}$ and $\theta_i^1, \theta_i^2 \in [0, 2\pi]$.

Parameters identical in all following simulations, in particular $\delta_2 > 1$, hence agent 2 has unstable dynamics (without coupling).

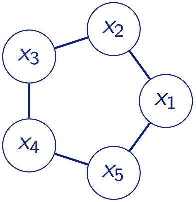
Simulation with constant k

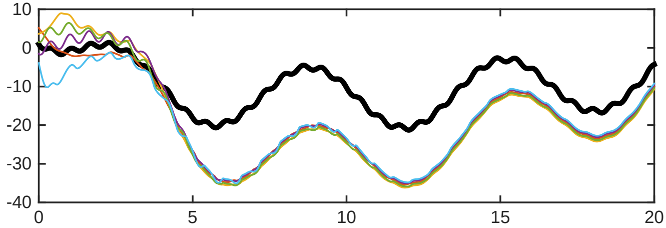
$u_i = -k\, e_i$ with $k = 10$





Funnel synchronization



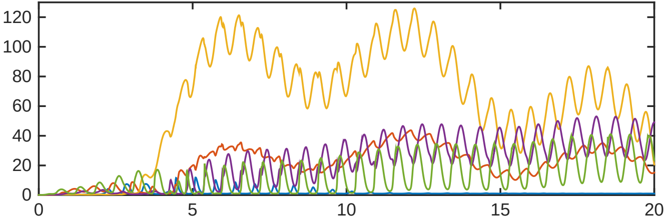


$$u_i(t) = -k_i(t)e_i(t)$$

$$k_i(t) = \frac{1}{\varphi(t) - |e_i(t)|}$$

$$\varphi(t) = \underline{\varphi} + (\overline{\varphi} - \underline{\varphi})e^{-\lambda t}$$

$$\overline{\varphi} = 20,\ \underline{\varphi} = 1,\ \lambda = 1$$



Observations for funnel synchronization from simulations

Funnel synchronization seems to work

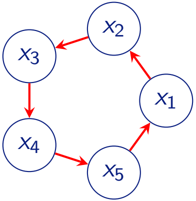
- errors remain within funnel
- practical synchronizations is achieved
- limit trajectory does not coincide with solution $s(\cdot)$ of

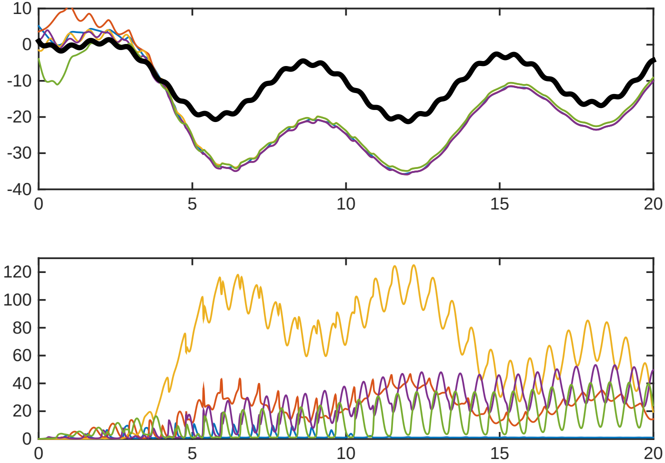
$$\dot{s}(t) = \frac{1}{N}\sum_{i=1}^{N} f_i(t, s(t)), \qquad s(0) = \frac{1}{N}\sum_{i=1}^{N} x_i.$$

What determines the new limiting trajectory?

- Coupling graph?
- Funnel shape?
- Gain function?

Funnel synchronization, directed graph





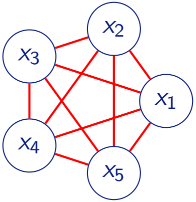
$$u_i(t) = -k_i(t)e_i(t)$$

$$k_i(t) = \frac{1}{\varphi(t) - |e_i(t)|}$$

$$\varphi(t) = \underline{\varphi} + (\overline{\varphi} - \underline{\varphi})e^{-\lambda t}$$

$$\overline{\varphi} = 20, \ \underline{\varphi} = 1, \ \lambda = 1$$

Funnel synchronization, complete graph

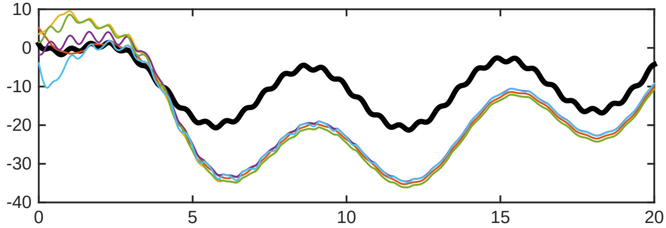


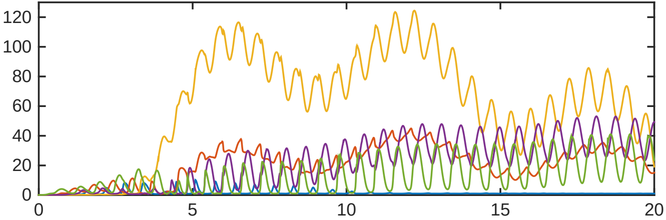
$$u_i(t) = -k_i(t)e_i(t)$$

$$k_i(t) = \frac{1}{\varphi(t) - |e_i(t)|}$$

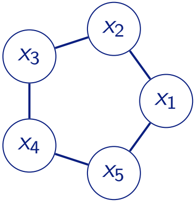
$$\varphi(t) = \underline{\varphi} + (\overline{\varphi} - \underline{\varphi})e^{-\lambda t}$$

$$\overline{\varphi} = 20,\ \underline{\varphi} = 1,\ \lambda = 1$$





Funnel synchronization with bigger funnel

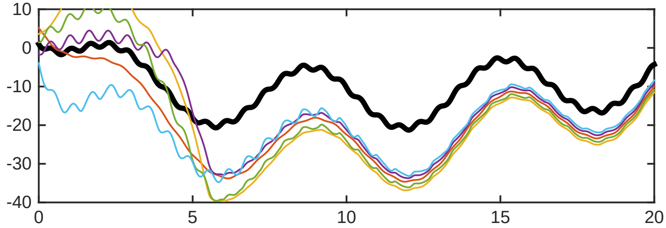


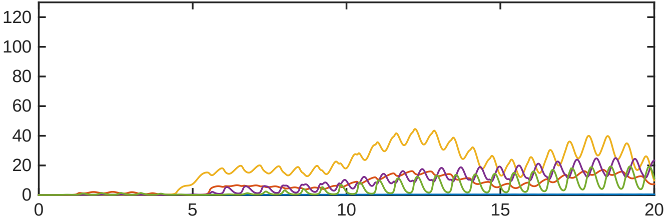
$$u_i(t) = -k_i(t)e_i(t)$$

$$k_i(t) = \frac{1}{\varphi(t) - |e_i(t)|}$$

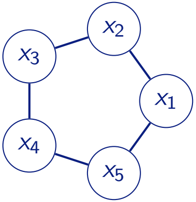
$$\varphi(t) = \underline{\varphi} + (\overline{\varphi} - \underline{\varphi})e^{-\lambda t}$$

$\overline{\varphi} = 30$, $\underline{\varphi} = 2$, $\lambda = 0.3$





Funnel synchronization with additional amplification

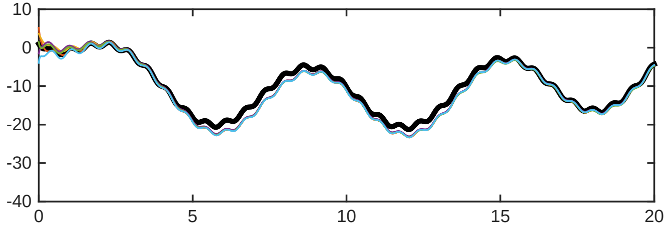


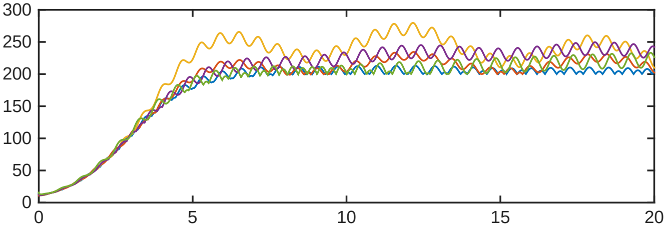
$$u_i(t) = -k_i(t)e_i(t)$$

$$k_i(t) = \frac{200}{\varphi(t) - |e_i(t)|}$$

$$\varphi(t) = \underline{\varphi} + (\overline{\varphi} - \underline{\varphi})e^{-\lambda t}$$

$$\overline{\varphi} = 20,\ \underline{\varphi} = 1,\ \lambda = 1$$





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Decentralized and weakly centralized Funnel synchronization

For fully decentralized Funnel synchronization

$$u_i(t) = -k_i(t)e_i(t) \quad \text{mit} \quad k_i(t) = \frac{1}{\varphi(t) - |e_i(t)|}$$

no theoretical results available yet.

Weakly centralized Funnel synchronization

Analogously as for diffusive coupling, all agents use the same gain:

$$u_i(t) = -k_{\max}(t)\, d_i\, e_i(t) \quad \text{with} \quad k_{\max}(t) := \max_{i \in V} \frac{1}{\varphi(t) - |e_i(t)|}$$

First theoretical result

Theorem

Assumption:

- *No „finite escape time“ of x_i*
- *The graph is connected, undirected and d-regular with*

$$d > \frac{N}{2} - 1$$

- *Funnel boundary $\varphi : [0, \infty) \to [\underline{\varphi}, \overline{\varphi}]$ is differentiable, non-increasing and*

$$|e_i(0)| < \varphi(0), \quad \forall i = 1, 2, \ldots, N.$$

Then weakly centralized funnel synchronization works.

Key arguments of the proof

Error dynamics for $e_1 := x_1 - \overline{x}_1$:

$$\begin{aligned}\dot{e}_1(t) &= f_1(t, x_1) - k_{\max}(t) d_1 e_1(t) - \dot{\overline{x}}_1(t) \\ &= \tilde{f}_1(t, x) - k_{\max}(t) d_1 e_1(t) + \frac{1}{d_1} \sum_{j \in \mathcal{N}_1} k_{\max}(t) d_j e_j(t)\end{aligned}$$

We need implication

$$\varphi(t) - |e_1(t)| \text{ small} \quad \Rightarrow \quad \dot{e}_1(t) \text{ large (with opposite sign as } e_1(t))$$

Because $k_{\max}(t) \geq \frac{1}{\varphi(t) - |e_1(t)|}$ we indeed have

$$\varphi(t) - |e_1(t)| \text{ small} \quad \Rightarrow \quad k_{\max}(t) \text{ large}$$

Key arguments of the proof

d-regularity assumption

$$\dot{e}_1(t) = \tilde{f}_1(t,x) - k_{\max}(t)\left(d\, e_1(t) - \sum_{j\in\mathcal{N}_i} e_j(t)\right)$$

Consequence of property of the Laplacian matrix:

$$\sum_{j\in\mathcal{N}_i} e_j = -e_1 - \sum_{j\neq\mathcal{N}_j\cup\{1\}} e_j$$

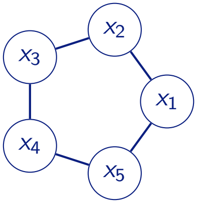
with

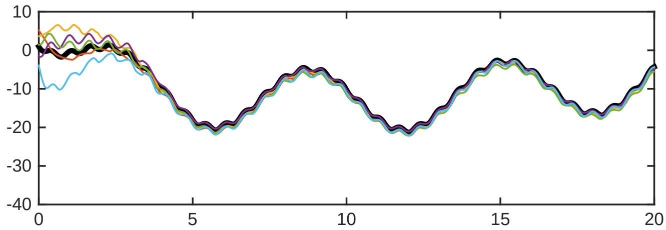
$$|\mathcal{N}_j \cup \{1\}| = N - d - 1$$

Hence, invoking $e_1(t) \geq e_j(t)$ for all $j \in V$,

$$\dot{e}_1(t) \leq \tilde{f}_1(t,x) - k_{\max}(t)(2d - N + 2)e_1(t).$$

Simulation weakly centralized Funnel synchronization





$$u_i(t) = -k_{\max}(t)e_i(t)$$

$$k_{\max}(t) = \max_{i\in V} \frac{1}{\varphi(t) - |e_i(t)|}$$

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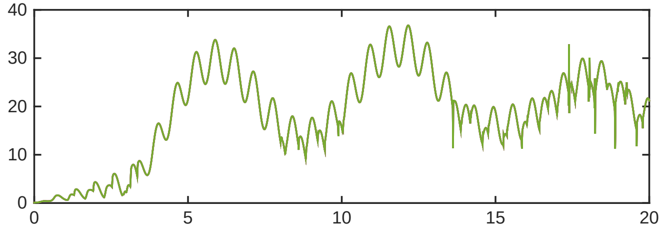
$$\overline{\varphi} = 20,\ \underline{\varphi} = 1,\ \lambda = 1$$



Summary

Combining diffusive coupling with funnel control leads to funnel synchronization

- local error feedback
- time-varying gain
- guaranteed transient behavior
- simulations look promising
- theoretical proof for weakly centralized funnel synchronization

Open questions

- limit trajectory
- weakly centralized case: non-regular graph or d small
- decentralized case