Controllability and observability are not dual for switched DAEs

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Naive dual of a switched DAE

Switched DAE

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u$$
$$y = C_\sigma x$$

Dual for switched DAE?

$$E_\sigma^\top \dot{p} = A_\sigma^\top p + C_\sigma^\top u_d$$
$$y_d = B_\sigma^\top p$$

Non-switched DAE

$$E\dot{x} = Ax + Bu$$
$$y = Cx$$

Classical dual [Cobb '84]

$$E^\top \dot{p} = A^\top p + C^\top u_d$$
$$y_d = B^\top p$$

An example

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u, \quad y = C_\sigma x$$

on $(-\infty, 1)$:	on $[1, 2)$:	on $[2, \infty)$:
$\dot{x}_1 = 0 + 0 \cdot u$	$\dot{x}_1 = 0 + 0 \cdot u$	$\dot{x}_1 = 0 + 0 \cdot u$
$0 = x_2$	$0 = x_1 - x_2$	$\dot{x}_2 = 0$
$y = 0$	$y = 0$	$y = x_2$

Solution

$$x_1(t) = x_1^0 \quad \forall t \in \mathbb{R}$$
$$x_2(t) = \mathbb{1}_{[1,\infty)}(t)\ x_1^0$$
$$y(t) = \mathbb{1}_{[2,\infty)}(t)\ x_1^0$$

$\Rightarrow$ **observable**

$$E_\sigma^\top \dot{p} = A_\sigma^\top p + C_\sigma^\top u_d, \quad y_d = B_\sigma^\top p$$

$\dot{p}_1 = 0 + 0 \cdot u_d$	$\dot{p}_1 = p_2 + 0 \cdot u_d$	$\dot{p}_1 = 0$
$0 = p_2$	$0 = -p_2$	$\dot{p}_2 = u_d$
$y_d = 0$	$y_d = 0$	$y_d = 0$

Solution

$$p_1(t) = p_1^0 \quad \forall t \in \mathbb{R}$$
$$p_2(t) = \mathbb{1}_{[2,\infty)} \int_2^t u_d$$

$\Rightarrow$ **not controllable**

Some remarks concerning duality

- Switched DAEs are special time-varying DAEs:

$$\begin{aligned} E(t)\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y &= C(t)x(t) \end{aligned}$$

whose dual is not (c.f. Balla & März '02, Kunkel & Mehrmann '08)

$$\begin{aligned} E(t)^\top \dot{p}(t) &= A(t)^\top p(t) + C(t)^\top u_d(t) \\ y_d &= B(t)^\top x(t) \end{aligned}$$

- For time-varying systems, adjoint system and dual system have to be distinguished, here:

$$\text{dual} \quad = \quad \text{time-inverted adjoint}$$

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Adjointness for linear ODEs

Linear ODE

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Adjoint of linear ODE

$$\dot{p} = -A^\top p - C^\top u_a$$
$$y_a = B^\top p$$

Input-State-Output-maps

- Input-map:
  $$u(\cdot) \mapsto g(\cdot) := Bu(\cdot)$$
- Input-state-map:
  $$(x_0, g(\cdot)) \mapsto (x(T), x(\cdot))$$
  $x(\cdot)$ solves $\dot{x} = Ax + g,\ x(0) = x_0$
- State-output-map:
  $$x(\cdot) \mapsto y(\cdot) := Cx(\cdot)$$

Adjoint maps

- Adjoint of input-map:
  $$p(\cdot) \mapsto y_a(\cdot) := B^\top p(\cdot)$$
- Adjoin of input-state-map:
  $$(p_T, h(\cdot)) \mapsto (p(0), p(\cdot))$$
  p solves $\dot{p} = -A^\top p - h,\ p(T) = p_T$
- Adjoint of state-output-map:
  $$u_a(\cdot) \mapsto h(\cdot) := C^\top u_a(\cdot)$$

Classical adjointness conditions

Behavior: $\mathcal{B}(A,B,C) := \{\ (u,x,y)\ \mid \dot{x} = Ax + Bu,\ \ y = Cx\ \}$

Theorem (van der Schaft '91)

(u_a, p, y_a) solves adjoint system $\Leftrightarrow$ following adjointness condition holds

$$\boxed{\tfrac{\mathrm{d}}{\mathrm{d}t}(p^\top x) - y_a^\top u + u_a^\top y = 0} \quad \forall (u,x,y) \in \mathcal{B}(A,B,C) \qquad (\mathbf{A})$$

In terms of behaviors:

$$\{\ (u_a, p, y_a)\ \mid (\mathbf{A})\ \text{holds}\ \} = \mathcal{B}(-A^\top, -C^\top, B^\top)$$

$\mathcal{B}(E(\cdot), A(\cdot)) := \{\ x\ \mid E(\cdot)\dot{x} = A(\cdot)x\ \}$

Adjointness condition for $E(t)\dot{x}(t) = A(t)x(t)$, Balla & März '02

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(p^\top E(\cdot)x) = 0, \quad \forall x \in \mathcal{B}(E(\cdot), A(\cdot))$$

Adjointness for switched DAEs

$\mathcal{B}(A,B,C) := \{ (u,x,y) \mid \dot{x} = Ax + Bu,\ y = Cx \}$ $\mathcal{B}(E,A) := \{ x \mid E\dot{x} = Ax \}$

Adjointness for $\dot{x} = Ax + Bu$, $y = Cx$

$$\forall (u,x,y) \in \mathcal{B}(A,B,C):$$
$$\tfrac{\mathrm{d}}{\mathrm{d}t}(p^\top x) - y_a^\top u + u_a^\top y = 0$$

Adj. for $E(\cdot)\dot{x} = A(\cdot)x$

$$\forall x \in \mathcal{B}(E(\cdot), A(\cdot)):$$
$$\tfrac{\mathrm{d}}{\mathrm{d}t}(p^\top E(\cdot)x) = 0,$$

Adjointness condition for switched DAEs and adjoint behavior

With $\mathcal{B}_\sigma := \{ (u,x,y) \mid E_\sigma \dot{x} = A_\sigma x + B_\sigma u,\ y = C_\sigma x \}$ let adjointness condition be:

$$\boxed{\tfrac{\mathrm{d}}{\mathrm{d}t}(p^\top E_\sigma x) - y_a^\top u + u_a^\top y = 0 \quad \forall (u,x,y) \in \mathcal{B}_\sigma} \qquad (\mathbf{A}_\sigma)$$

Furthermore, a behavior $\mathcal{B} \subseteq \{(u_a, p, y_a)\}$ is called a behavioral adjoint of $\mathcal{B}_\sigma$ $:\Leftrightarrow$

$$(\mathbf{A}_\sigma) \text{ holds } \forall (u,x,y) \in \mathcal{B}_\sigma$$

Behavioral adjoint representation

Theorem

Consider

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(p^\top E_\sigma) = -p^\top A_\sigma - u_a^\top C_\sigma, \qquad \textbf{(adj)}$$
$$y_a^\top = p^\top B_\sigma$$

Then

$$\mathcal{B}^a_\sigma := \{\ (u_a, p, y_a)\ \mid (u_a, p, y_a) \text{ satisfies } \textbf{(adj)}\ \}$$

is a behavioral adjoint of $\mathcal{B}_\sigma$*.*

Attention

- Switched DAE and **(adj)** are equations in a certain distribution space
- In this space only non-commutative multiplication is defined, in particular
$$p^\top A_\sigma \neq (A_\sigma^\top p)^\top$$
- **(adj)** is not causal
- Piecewise-constant E_σ is differentiated $\rightarrow$ Dirac impulses occur in coefficient matrices

Problem: Adjoint is not a switched DAE

Fundamental problem

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}(p^\top E_\sigma) &= -p^\top A_\sigma - u_a^\top C_\sigma, \\ y_a^\top &= p^\top B_\sigma \end{aligned} \qquad \textbf{(adj)}$$

is not a switched DAE, in particular:

- Solution theory?
- Controllability, observability?

Time-inversion

Problems can be resolved by considering time-inversion and recalling

dual = time-inverted adjoint

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Time-inversion and T-dual

Definition (Time inversion for distributions)

For $T \in \mathbb{R}$ let $\mathscr{T}_T : \mathbb{D} \to \mathbb{D}$ denote the time-inversion at T on the space of distributions $\mathbb{D}$, i.e. for all test functions $\varphi \in \mathcal{C}_0^\infty$ and all distributions $D \in \mathbb{D}$:

$$\mathscr{T}_T(D)(\varphi) := D(\varphi(T - \cdot))$$

Convention: $s = T - t$ and $\widetilde{\sigma} := \sigma(T - \cdot)$

Definition (T-dual of switched DAE)

Let $\mathcal{B}_\sigma^a$ be a behavioral adjoint of switched DAE. The T-dual behavior of the switched DAE is

$$\mathcal{B}_\sigma^{T\text{-dual}} := \{ \ (u_d, z, y_d) \ | \ (u_a, p, y_a) = (\mathscr{T}_T(u_d), \mathscr{T}_T(z), \mathscr{T}_T(y_d)) \in \mathcal{B}_\sigma^a \ \}$$

Question

Representable as switched DAE?

Theorem (Switched DAE representation of T-dual)

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}s}(E_{\widetilde{\sigma}}^\top z) &= A_{\widetilde{\sigma}}^\top z + C_{\widetilde{\sigma}}^\top u_d \\ y_d &= B_{\widetilde{\sigma}}^\top y \end{aligned} \qquad \textbf{(dual)}$$

is a T-dual of switched DAE.

Almost a switched DAE:

$$\textbf{(dual)} \quad \Leftrightarrow \quad \begin{aligned} E_{\widetilde{\sigma}}^\top \dot{z} &= A_{\widetilde{\sigma}}^\top z + C_{\widetilde{\sigma}}^\top u_d - (\tfrac{\mathrm{d}}{\mathrm{d}s} E_{\widetilde{\sigma}}^\top) z \\ y_d &= B_{\widetilde{\sigma}}^\top y \end{aligned}$$

where

$$\tfrac{\mathrm{d}}{\mathrm{d}s} E_{\widetilde{\sigma}}^\top = \sum_i (E_{i-1} - E_i)^\top \delta_{T-t_i}$$

$\Rightarrow$ New system class: Switched DAEs with impacts (c.f. T. & Willems '12)

Switched DAEs with impacts and their dual

Sw. DAEs with impacts

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u + G[\cdot]x$$
$$y = C_\sigma x$$

Dual (via time-inversion of adjoint)

$$E_{\tilde{\sigma}}^\top \dot{z} = A_{\tilde{\sigma}}^\top z + C_{\tilde{\sigma}}^\top u_d + (\mathscr{T}_T(G[\cdot])^\top - \tfrac{\mathrm{d}}{\mathrm{d}s} E_{\tilde{\sigma}}^\top) z$$
$$y = B_{\tilde{\sigma}}^\top x$$

where, for the switching times t_i of σ, $G[\cdot] := \sum_i G_{t_i} \delta_{t_i}$

Theorem (Dual of dual)

If σ is constant outside of $(0, T)$, then the T-dual of the T-dual is the original switched DAE with impacts.

Crucial ingredients

- Suitable adjointness condition
- Time-inversion
- Extension of system class: Switched DAEs with impacts

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Observability and Determinability

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u + G[\cdot]x$$
$$y = C_\sigma x$$

(swDAE+i)

Definition (Observability)

(**swDAE+i**) is called observable on $[0, T]$ $:\Leftrightarrow$ the following implication holds for all solutions

$$u = 0 \ \wedge \ y_{[0,T]} = 0 \quad \Rightarrow \quad x = 0$$

Definition (Determinability)

(**swDAE+i**) is called determinable on $[0, T]$ $:\Leftrightarrow$ the following implication holds for all solutions

$$u = 0 \ \wedge \ y_{[0,T]} = 0 \quad \Rightarrow \quad x_{(T,\infty)} = 0$$

Obviously, observability $\Rightarrow$ determinability
But the converse is not true in general

Controllability and Reachability

$$\mathcal{B}_\sigma := \{\ (u,x,y)\ \mid E_\sigma \dot{x} = A_\sigma x + B_\sigma u + G[\cdot]x,\ \ y = C_\sigma x\ \}$$

Definition (Controllability)

(**swDAE+i**) is called controllable on $[0,T]$:⇔

$$\forall w = (u,x,y) \in \mathcal{B}_\sigma\ \exists \widetilde{w} = (\widetilde{u},\widetilde{x},\widetilde{y}) \in \mathcal{B}_\sigma :$$
$$w_{(-\infty,0)} = \widetilde{w}_{(-\infty,0)}\ \wedge\ \widetilde{w}_{(T,\infty)} = 0$$

Definition (Reachability)

(**swDAE+i**) is called reachable on $[0,T]$:⇔

$$\forall w = (u,x,y) \in \mathcal{B}_{\sigma(T^+)}\ \exists \widetilde{w} = (\widetilde{u},\widetilde{x},\widetilde{y}) \in \mathcal{B}_\sigma :$$
$$\widetilde{w}_{(-\infty,0)} = 0\ \wedge\ \widetilde{w}_{(T,\infty)} = w_{(T,\infty)}$$

Easily seen: reachability ⇒ controllability
But the converse is not true in general

Main Duality result

Theorem

For switched DAE with impacts it holds that

$$\begin{array}{ccc} \textit{Observability} & \overset{\textit{dual}}{\leftrightsquigarrow} & \textit{Reachability} \\ \Downarrow & & \Downarrow \\ \textit{Determinability} & \overset{\textit{dual}}{\leftrightsquigarrow} & \textit{Controllability.} \end{array}$$

Proof is based on some recent observability/determinability (Tanwani & T. '12) and controllability/reachability characterizations (Ruppert, Küsters & T. '15) for switched DAEs