Controllability characterization for switched DAEs

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Switched DAEs

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t)$$

or short (and actually more suitable)

$$\boxed{E_\sigma \dot{x} = A_\sigma x + B_\sigma u} \qquad \textbf{(swDAE)}$$

Assumptions:

- switching signal $\sigma : \mathbb{R} \to \mathcal{P}$ piecewise-constant
  in particular, no accumulation of switching times
- each matrix pair (E_p, A_p), $p \in \mathcal{P}$, is regular, i.e. $\det(sE_p - A_p) \not\equiv 0$
- piecewise-smooth distributional solution framework [T. 2009]
  i.e. $x \in \mathbb{D}^n_{\text{pw}\mathcal{C}^\infty}$, $u \in \mathbb{D}^m_{\text{pw}\mathcal{C}^\infty}$

$$\mathbb{D}_{\text{pw}\mathcal{C}^\infty} = \left\{ D = f_{\mathbb{D}} + \sum_{t \in T} D_t \;\middle|\; \begin{array}{l} f \text{ is piecewise smooth, } T \subseteq \mathbb{R} \text{ discrete} \\ \forall t \in T : D_t \in \operatorname{span}\{\delta_t, \delta_t', \delta_t'', \ldots\} \end{array} \right\}$$

Controllability in the behavioral sense

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u \qquad \textbf{(swDAE)}$$

Definition (Distributional solution behavior)

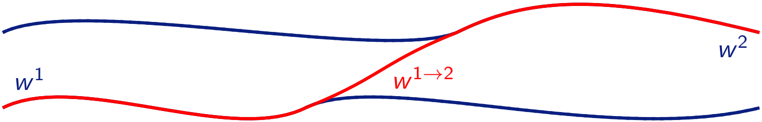
$$\mathcal{B}_\sigma := \left\{ w := (x,u) \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}^{n+m} \;\middle|\; E_\sigma \dot{x} = A_\sigma x + B_\sigma u \right\}$$

Definition (Controllability (from $t = 0$))

(swDAE) controllable $:\Leftrightarrow$ $\mathcal{B}_\sigma$ is controllable, i.e.

$$\forall w^1, w^2 \in \mathcal{B}_\sigma \; \exists T \geq 0 \; \exists w^{1\to 2} \in \mathcal{B}_\sigma :$$

$$w^{1\to 2}_{(-\infty,0)} = w^1_{(-\infty,0)} \quad \wedge \quad w^{1\to 2}_{(T,\infty)} = w^2_{(T,\infty)}$$



Comments on controllability definition

Lemma (Controllability to origin)

(swDAE) *controllable* $\Leftrightarrow$

$$\forall w \in \mathcal{B}_\sigma \ \exists T \geq 0 \ \exists w^0 \in \mathcal{B}_\sigma : \quad w^0_{(-\infty,0)} = w_{(0,\infty)} \ \wedge \ w^0_{(T,\infty)} = 0$$

Definition (Controllability subspaces)

$$\mathcal{C}_\sigma^{[t_0,t_1]} := \left\{ \ x_0 \in \mathbb{R}^n \ \middle| \ \exists (x,u) \in \mathcal{B}_\sigma : x(t_0^-) = x_0 \ \wedge x(t_1^+) = 0 \ \right\}$$

Feasibility of initial values

$\mathcal{F}_\sigma^{t^-} := \{ \ x(t^-) \ | \ (x,u) \in \mathcal{B}_\sigma \ \} \neq \mathbb{R}^n$ (in general)

(swDAE) controllable $\not\Rightarrow$ $\mathcal{C}_\sigma^{[0,T]} = \mathbb{R}^n$

(swDAE) controllable $\Leftrightarrow$ $\mathcal{C}_\sigma^{[0,T]} = \mathcal{F}_\sigma^{0^-}$

Note: $\mathcal{F}_\sigma^{t^-} \neq \mathcal{F}_{\sigma(t^-)}^{t^-} =: \mathcal{F}_{\sigma(t^-)}$ in general!

Illustrative example

mode -1	mode 0	mode 1
$(-\infty, 0)$	$[0, t_1)$	$[t_1, \infty)$
$\dot{x}_1 = 0$ $x_2 = 0$	$\dot{x}_1 = 0$ $x_2 = u$	$\dot{x}_1 = u$ $\dot{x}_2 = u$
$\mathcal{C}_{-1} = \{0\}$	$\mathcal{C}_0 = \operatorname{im} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\mathcal{C}_1 = \operatorname{im} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\mathcal{F}_{-1} = \operatorname{im} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\mathcal{F}_0 = \mathbb{R}^2$	$\mathcal{F}_1 = \mathbb{R}^2$

Controllability space

$$\mathcal{C}_\sigma^{[0,t_1]} = \mathcal{C}_\sigma^{[0,t_1+\varepsilon]} = \operatorname{im} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathcal{F}_{-1} \quad \Rightarrow \quad \text{controllable}$$

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Regular DAEs and the quasi-Weierstrass form

Theorem ((Quasi-)Weierstrass form, [Weierstrass 1868])

(E, A) *is regular*

$$\Leftrightarrow \exists S, T \text{ invertible: } (SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \; N \text{ nilpotent}$$

Calculate S, T via Wong-sequences [Wong 1974; Berger, Ilchmann, T. 2012]

Definition (Some useful "projectors")

Consistency projector: $\Pi := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$

Differential projector: $\Pi^{\text{diff}} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S$

Impulse projector: $\Pi^{\text{imp}} := T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S$

$$A^{\text{diff}} := \Pi^{\text{diff}} A, \quad B^{\text{diff}} := \Pi^{\text{diff}} B, \quad E^{\text{imp}} := \Pi^{\text{imp}} E, \quad B^{\text{imp}} := \Pi^{\text{imp}} B$$

Controllability characterization (unswitched case)

Theorem (T. 2012)

(x,u) smooth solution of (E,A) $\Leftrightarrow \exists x^0 \in \mathbb{R}^n\ \forall t \in \mathbb{R}$:

$$x(t) = e^{A^{\text{diff}}t}\Pi x^0 + \int_0^t e^{A^{\text{diff}}(t-s)} B^{\text{diff}} u(s)\,\mathrm{d}s - \sum_{i=0}^{n-1} (E^{\text{imp}})^i B^{\text{imp}} u^{(i)}(t)$$

Corollary (Feasibility and controllability spaces)

Feasibility space: $\mathcal{F} = \operatorname{im}\Pi \oplus \langle E^{\text{imp}}, B^{\text{imp}}\rangle$

Controllability space: $\mathcal{C} = \langle A^{\text{diff}}, B^{\text{diff}}\rangle \oplus \langle E^{\text{imp}}, B^{\text{imp}}\rangle$

where $\langle A, B\rangle := \operatorname{im}[B, AB, A^2B, \ldots, A^{n-1}B]$

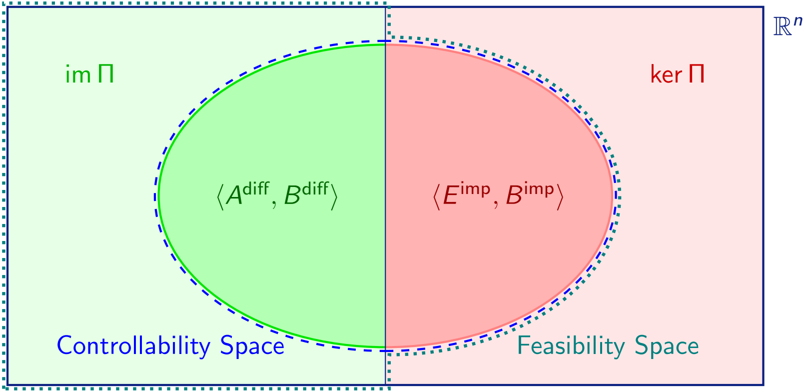
Corollary (Controllability characterization)

$E\dot{x} = Ax + Bu$ *controllable (in the behavioral sense)*

$\Leftrightarrow \quad \langle A^{\text{diff}}, B^{\text{diff}}\rangle = \operatorname{im}\Pi$

$\Leftrightarrow \quad \langle A^{\text{diff}}, B^{\text{diff}}\rangle + \ker\Pi = \mathbb{R}^n$

Overall picture



Attention

Controllability independent of $\langle E^{\text{imp}}, B^{\text{imp}}\rangle$, but the latter essential in switched case (previous example)

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Recursive formula for controllability space

$$\boxed{E_\sigma \dot{x} = A_\sigma x + B_\sigma u} \qquad \textbf{(swDAE)}$$

with switching signal $(0 =: t_0 < t_1 < t_2 < \dots)$ and its restriction $(s \geq 0)$

$$\sigma(t) := \begin{cases} -1, & t < 0, \\ i, & t \in [t_i, t_{i+1}), \end{cases} \qquad \sigma_{\geq s}(t) := \begin{cases} \sigma(s^+), & t \leq s, \\ \sigma(t), & t \geq s \end{cases}$$

Theorem (Controllability recursion, KRT 2015)

$$\mathcal{C}_{\sigma_{\geq t_\ell}}^{[t_\ell, t_\ell]} = \mathcal{C}_{\sigma_{\geq t_\ell}}^{[t_\ell, t_\ell+\varepsilon]} = \mathcal{C}_\ell, \quad \ell \in \mathbb{N}$$

$$\mathcal{C}_{\sigma_{\geq t_{k-1}}}^{[t_{k-1}, \ell]} = \left(\mathcal{C}_{k-1} + e^{-A_{k-1}^{\text{diff}}(t_k - t_{k-1})} \Pi_k^{-1} \mathcal{C}_{\sigma_{\geq t_k}}^{[t_k, t_\ell]} \right) \cap \mathcal{F}_{k-1}, \quad 0 < k \leq \ell$$

$$\mathcal{C}_{\sigma}^{[0, t_\ell]} = \Pi_0^{-1} \mathcal{C}_{\sigma_{\geq 0}}^{[0, t_\ell]} \cap \mathcal{F}_{-1} \overset{!}{=} \mathcal{F}_{-1}$$

Conclusions

- Controllability of switched DAEs
  - Distributional solution theory (jumps and Dirac impulses)
  - Controllability in the behavioral sense
  - Recursion formula for controllability space
  - Several pitfalls on the way
    - $\langle E^{\text{imp}}, B^{\text{imp}} \rangle$ irrelevant for unswitched controllability, but essential for switched case, in particular

$$\langle A_0^{\text{diff}}, B_0^{\text{diff}} \rangle + \Pi_1^{-1} \langle A_1^{\text{diff}}, B_1^{\text{diff}} \rangle \supseteq \mathcal{F}_0$$

$$\not\Leftrightarrow$$

$$\langle A_0^{\text{diff}}, B_0^{\text{diff}} \rangle \oplus \langle E_0^{\text{imp}}, B_0^{\text{imp}} \rangle + \Pi_1^{-1} \langle A_1^{\text{diff}}, B_1^{\text{diff}} \rangle \supseteq \mathcal{F}_0 \oplus \langle E^{\text{imp}}, B_0^{\text{imp}} \rangle$$

$$\not\Leftrightarrow$$

$$\langle A_0^{\text{diff}}, B_0^{\text{diff}} \rangle \oplus \ker \Pi_0 + \Pi_1^{-1} \langle A_1^{\text{diff}}, B_1^{\text{diff}} \rangle \supseteq \mathbb{R}^n$$

    - $\mathcal{F}_\sigma^{t^-} \neq \mathcal{F}_{\sigma(t-)}$
- Further topics
  - Duality with observability ✓
  - Controllability of Dirac impulses ?
  - Control via switching signal ?