Basics on Differential-Algebraic Equations (DAEs)

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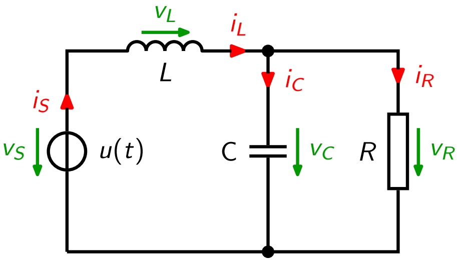
Modeling of electrical circuits



Basic circuit elements

Resistor: $v_R(t) = R\, i_R(t)$

Capacitor: $i_C(t) = C \frac{\mathrm{d}}{\mathrm{d}t} v_C(t)$

Inductor: $v_L(t) = L \frac{\mathrm{d}}{\mathrm{d}t} i_L(t)$

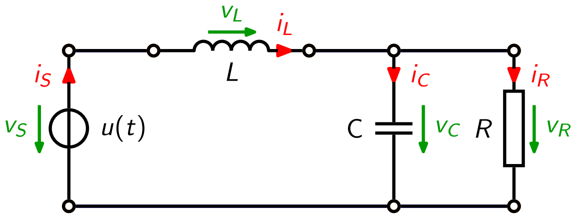
Voltage source: $v_S(t) = u(t)$

DAEs

All components are given by a differential-algebraic equation (DAE)

$$E\dot{x} = Ax + Bu$$

Hierarchical model building



Overall model
⇒ Again DAE:

$$E\dot{x} = Ax + Bu$$

$$\begin{bmatrix} 0 & 0 & & & & & & \\ & & C & 0 & & & & \\ & & & & 0 & L & & \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \\ & & & & & & & 0 \\ & & & & & & & 0 \\ & & & & & & & 0 \end{bmatrix} \begin{pmatrix} \dot{v}_R \\ \dot{i}_R \\ \dot{v}_C \\ \dot{i}_C \\ \dot{v}_L \\ \dot{i}_L \\ \dot{v}_S \\ \dot{i}_S \end{pmatrix} = \begin{bmatrix} -1 & R & & & & & & \\ & & 0 & 1 & & & & \\ & & & & 1 & 0 & & \\ & & & & & & -1 & 0 \\ 1 & & -1 & & & & & \\ & 1 & & 1 & & -1 & & \\ & & 1 & & 1 & & -1 & \\ & & & & & 1 & & -1 \end{bmatrix} \begin{pmatrix} v_R \\ i_R \\ v_C \\ i_C \\ v_L \\ i_L \\ v_S \\ i_S \end{pmatrix} + \begin{bmatrix} \\ \\ \\ 1 \\ \\ \\ \\ \\ \end{bmatrix} u$$

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Recall ODEs

Ordinary differential equations (ODEs):

$$\dot{x} = Ax + f$$

- Initial values: arbitrary
- Solution uniquely determined by f and $x(0)$
- No inhomogeneity constraints

Simple DAE example

DAE example:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

$$\begin{aligned} \dot{x}_2 &= x_1 + f_1 \longrightarrow x_1 = -f_1 - \dot{f}_2 \\ 0 &= x_2 + f_2 \longrightarrow x_2 = -f_2 \nearrow \\ 0 &= f_3 \end{aligned}$$

no restriction on x_3

Conclusions from example

Solution of example:

$$x_1 = -f_1 - \dot{f}_2$$
$$x_2 = -f_2$$
$$x_3 \text{ free}$$

$$f_3 = 0 \text{ necessary}$$

Differences to ODEs

- For fixed inhomogeneity, initial values cannot be chosen arbitrarily ($x_1(0) = -f_1(0) - \dot{f}_2(0)$, $x_2(0) = f_2(0)$)
- For fixed inhomogeneity, solution not uniquely determined by initial value (x_3 free)
- Inhomogeneity not arbitrary
  - structural restrictions ($f_3 = 0$)
  - differentiability restrictions ($\frac{\mathrm{d}}{\mathrm{d}t} f_2$ must be well defined)

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Nilpotent DAEs

$$\begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \dot{x} = x + f$$

$$\begin{aligned}
\Leftrightarrow \quad 0 &= x_1 + f_1 & \longrightarrow \quad x_1 &= -f_1 \\
\dot{x}_1 &= x_2 + f_2 & \longrightarrow \quad x_2 &= -f_2 - \dot{f}_1 \\
\dot{x}_2 &= x_3 + f_3 & \longrightarrow \quad x_3 &= -f_3 - \dot{f}_2 - \ddot{f}_1 \\
&\vdots & \vdots \qquad & \vdots \\
\dot{x}_{n-1} &= x_n + f_n & \longrightarrow \quad x_n &= -\sum_{i=1}^{n} f_i^{(n-i)}
\end{aligned}$$

General nilpotent DAE

In general: $N\dot{x} = x + f$ with N nilpotent, i.e. $N^n = 0$

$$\overset{N\frac{\mathrm{d}}{\mathrm{d}t}}{\Rightarrow} N^2\ddot{x} = N\dot{x} + N\dot{f} = x + f + N\dot{f}$$

$$\overset{N\frac{\mathrm{d}}{\mathrm{d}t}}{\Rightarrow} N^3\dddot{x} = N^2\ddot{x} + N^2\ddot{f} = x + f + N\dot{f} + N^2\ddot{f}$$

$$\vdots$$

$$\overset{N\frac{\mathrm{d}}{\mathrm{d}t}}{\Rightarrow} \underbrace{N^n x^{(n)}}_{=0} = x + \sum_{i=0}^{n-1} N^i f^{(i)} \quad \Rightarrow \quad \boxed{x = -\sum_{i=0}^{n-1} N^i f^{(i)}}$$

Properties

- Initial values: fixed by inhomogeneity
- Solution uniquely determined by f
- Inhomogeneity constraints:
  - no structural constraints
  - differentiability constraints: $\sum_{i=0}^{n-1} N^i f^{(i)}$ needs to be well defined

Underdetermined DAEs

$$\underbrace{\begin{bmatrix} 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & 1 & 0 & \\ & & & 1 & 0 \end{bmatrix}}_{n-1 \times n} \dot{x} = \begin{bmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 & \\ & & & 0 & 1 \end{bmatrix} x + f$$

$$\Leftrightarrow \begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_{n-2} \\ \dot{x}_{n-1} \end{pmatrix} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_n \end{pmatrix} + f$$

$\Leftrightarrow$ ODE with additional "input" x_n

Properties

- Initial values: arbitrary
- Solution not uniquely determined by $x(0)$ and f
- Inhomogeneity constraints: none

Overdetermined DAEs

$$
n+1 \overbrace{\begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & 0 & \\ & & 1 & 0 \\ & & & \color{red}{1} \end{bmatrix}}^{n} \dot{x} = \begin{bmatrix} 1 & & & \\ 0 & \ddots & & \\ & \ddots & 1 & \\ & & 0 & 1 \\ & & & \color{red}{0} \end{bmatrix} x + f
$$

$$
\Leftrightarrow \underbrace{\begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & 0 & \\ & & 1 & 0 \end{bmatrix}}_{=:N} \dot{x} = x + \begin{pmatrix} f_1 \\ \vdots \\ f_{n-1} \\ f_n \end{pmatrix} \;\wedge\; \color{red}{\dot{x}_n = f_{n+1}}
$$

$$
\Leftrightarrow x = -\sum_{i=0}^{n-1} N^i f^{(i)} \;\wedge\; \underbrace{\dot{x}_n = -\sum_{i=1}^{n} f_i^{n-i+1} \overset{!}{=} f_{n+1}}_{\Leftrightarrow\, \color{red}{\sum_{i=1}^{n+1} f_i^{(n+1-i)} = 0}}
$$

Overdetermined DAEs properties

$$x = -\sum_{i=0}^{n-1} N^i f^{(i)} \;\wedge\; \sum_{i=1}^{n+1} f_i^{(n+1-i)} = 0$$

Properties

- Initial valus: fixed by inhomogeneity
- Solution uniquely determined by f
- Inhomogeneity constraints
  - structural constraint: $\sum_{i=1}^{n+1} f_i^{(n+1-i)} = 0$
  - differentiability constraint: $f_i^{(n+1-i)}$ needs to be well defined

No other cases

All DAEs are combinations of
ODEs, nilpotent DAEs, underdetermined DAEs, overdetermined DAEs

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Equivalence

Fact 1

For any invertible matrix $S \in \mathbb{R}^{m\times m}$:

$$(x,u) \text{ solves } E\dot{x} = Ax + Bu \;\Leftrightarrow\; (x,u) \text{ solves } SE\dot{x} = SAx + SBu$$

Fact 2

For coordinate transformation $T \in \mathbb{R}^{n\times n}$:

$$(x,u) \text{ solves } E\dot{x} = Ax + Bu \;\overset{x=Tz}{\Longleftrightarrow}\; (z,u) \text{ solves } ET\dot{z} = ATz + Bu$$

Together

$$(x,u) \text{ solves } E\dot{x} = Ax + Bu \;\overset{x=Tz}{\Longleftrightarrow}\; (z,u) \text{ solves } SET\dot{z} = SATz + SBu$$

Definition

(E_1, A_1), (E_2, A_2) *equivalent* $:\Leftrightarrow\; (E_2, A_2) = (SE_1T, SA_1T)$, short:

$$(E_1, A_1) \cong (E_2, A_2)$$

Quasi-Kronecker form (QKF)

Theorem (Quasi-Kronecker Form)

For any $E, A \in \mathbb{R}^{\ell \times m}$

$$(E,A) \cong \left(\begin{bmatrix} E_U & & & \\ & I & & \\ & & N & \\ & & & E_O \end{bmatrix}, \begin{bmatrix} A_U & & & \\ & J & & \\ & & I & \\ & & & A_O \end{bmatrix} \right)$$

where (E_U, A_U) consists of underdetermined blocks on the diagonal, N is nilpotent, and (E_O, A_O) consists of overdetermined diagonal blocks

QKF Examples

Remark

0×1 and 1×0 underdetermined/overdetermined blocks are possible

Example: $$\left(\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\right)\cong\left(\begin{bmatrix} \overline{} & & \\ & \begin{bmatrix}0&1\\0&0\end{bmatrix} & \\ & & \| \end{bmatrix},\begin{bmatrix} \overline{} & & \\ & \begin{bmatrix}1&0\\0&1\end{bmatrix} & \\ & & \| \end{bmatrix}\right)$$

(E,A) from circuit $$\cong\left(\begin{bmatrix} I_{2\times 2} & \\ & 0_{6\times 6}\end{bmatrix},\begin{bmatrix} \begin{matrix}0 & {}^1/_C\\ {}^{-1}/_L & {}^{-1}/_{RC}\end{matrix} & \\ & I_{6\times 6}\end{bmatrix}\right)$$

Regularity

$$(E,A) \cong \left(\begin{bmatrix} \xcancel{E_U} & & & \\ & I & & \\ & & N & \\ & & & \xcancel{E_O} \end{bmatrix}, \begin{bmatrix} \xcancel{A_U} & & & \\ & J & & \\ & & I & \\ & & & \xcancel{A_O} \end{bmatrix} \right)$$

Corollary (Quasi-Weierstrass-Form (QWF))

$E\dot{x} = Ax + f$ has solution x for any sufficiently smooth f and each solution x is uniquely determined by $x(0)$ and f

$$\Leftrightarrow \quad (E,A) \cong \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right)$$

quasi-Weierstrass form

(E,A) is then called **regular** ($\Leftrightarrow$ $\det(sE - A)$ not the zero polynomial).

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Definition of Wong sequences

Definition

Let $E, A \in \mathbb{R}^{m\times n}$. The corresponding Wong sequences of the pair (E, A) are:

$$\mathcal{V}_0 := \mathbb{R}^n, \qquad \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i), \qquad i = 0, 1, 2, 3, \ldots$$
$$\mathcal{W}_0 := \{0\}, \qquad \mathcal{W}_{j+1} := E^{-1}A(\mathcal{W}_j), \qquad j = 0, 1, 2, 3, \ldots$$

Note: $M^{-1}\mathcal{S} := \{\ x \mid Mx \in \mathcal{S}\ \}$ and $M\mathcal{S} := \{\ Mx \mid x \in \mathcal{S}\ \}$

Clearly, $\exists i^*, j^* \in \mathbb{N}$

$$\mathcal{V}_0 \supset \mathcal{V}_1 \supset \ldots \supset \mathcal{V}_{i^*} = \mathcal{V}_{i^*+1} = \mathcal{V}_{i^*+2} = \ldots$$
$$\mathcal{W}_0 \subset \mathcal{W}_1 \subset \ldots \subset \mathcal{W}_{j^*} = \mathcal{W}_{j^*+1} = \mathcal{W}_{j^*+2} = \ldots$$

Wong limits:

$$\boxed{\mathcal{V}^* := \bigcap_{i\in\mathbb{N}} \mathcal{V}_i = \mathcal{V}_{i^*}} \qquad \boxed{\mathcal{W}^* = \bigcup_{i\in\mathbb{N}} \mathcal{W}_i = \mathcal{W}_{j^*}}$$

Wong sequences and the QWF

Theorem

The following statements are equivalent for square $E, A \in \mathbb{R}^{n\times n}$:

(i) (E, A) *is regular*

(ii) $\mathcal{V}^* \oplus \mathcal{W}^* = \mathbb{R}^n$

(iii) $E\mathcal{V}^* \oplus A\mathcal{W}^* = \mathbb{R}^n$

In particular, with $\operatorname{im} V = \mathcal{V}^*$, $\operatorname{im} W = \mathcal{W}^*$

$$(E, A) \text{ regular} \quad \Rightarrow \quad T := [V, W] \text{ and } S := [EV, AW]^{-1} \text{ invertible}$$

and S, T *yield QWF:*

$$(SET, SAT) = \left(\begin{bmatrix} I & \\ & N \end{bmatrix}, \begin{bmatrix} J & \\ & I \end{bmatrix} \right), \quad N \text{ nilpotent}$$

Calculation of Wong sequences

Remark

Wong sequences can easily be calculated with Matlab even when the matrices still contain symbolic entries (like "R", "L", "C").

```
function V=getPreImage(A,S)
% returns a basis of the preimage of A of the linear space spanned by
% the columns of S, i.e. im V = { x | Ax \in im S }

[m1,n1]=size(A); [m2,n2]=size(S);
if m1==m2
    H=null([A,S]);
    V=colspace(H(1:n1,:));
else
    error('Both matrices must have same number of rows');
end;
```

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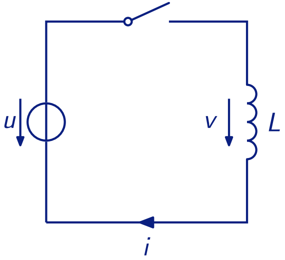
Circuit example



open switch: $0 = i$,

inductivity law: $L\frac{d}{dt}i = v$

Nilpotent DAE model

$$\begin{bmatrix} 0 & 0 \\ L & 0 \end{bmatrix} \dot{x} = x, \quad x = \begin{pmatrix} i \\ v \end{pmatrix}$$

$\Rightarrow$ unique solution $x(t) = 0 \; \forall t$ for which switch is open

Now assume switch was closed for $t < 0$

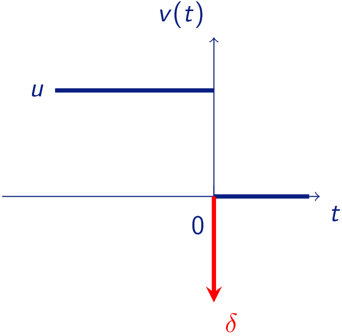
$\Rightarrow$ Different DAE-model for $t < 0$

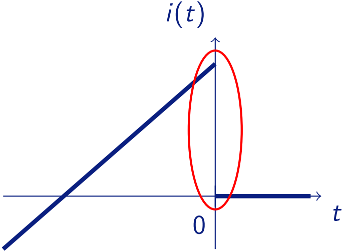
$\Rightarrow$ Inconsistent initial values for above DAE

Solution of circuit example

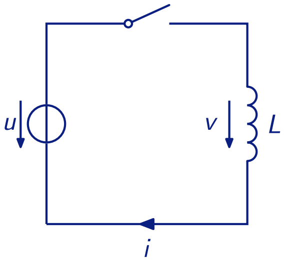
$t < 0$	$t \geq 0$
$v = u$	$i = 0$
$L\frac{\mathrm{d}}{\mathrm{d}t}i = v$	$v = L\frac{\mathrm{d}}{\mathrm{d}t}i$

Solution (assume constant input u):





Observations



Observations

- $x(0^-) \neq 0$ inconsistent for $\begin{bmatrix} 0 & 0 \\ L & 0 \end{bmatrix} \dot{x} = x$
- unique jump from $x(0^-)$ to $x(0^+)$
- derivative of jump = Dirac impulse appears in solution

Initial trajectory problem

Definition (Initial trajectory problem (ITP))

Given past trajectory $x^0 : (-\infty, 0) \to \mathbb{R}^n$ find $x : \mathbb{R} \to \mathbb{R}^n$ such that

$$\left.\begin{aligned} x\big|_{(-\infty,0)} &= x^0 \\ (E\dot{x})\big|_{[0,\infty)} &= (Ax+f)\big|_{[0,\infty)} \end{aligned}\right\} \qquad \text{(ITP)}$$

"Theorem" (Unique jump rule)

Consider (ITP) with $f = 0$ and regular (E, A) with QWF

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right).$$

Then any solution x of (ITP) satisfies

$$\boxed{x(0^+) = \Pi_{(E,A)} x(0^-)} \qquad \text{where } \Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$$

is the consistency projector.

Proof of unique jump rule "Theorem"

Let $\binom{v}{w} = T^{-1}x$, then x solves (ITP) with $f = 0$ $\Leftrightarrow$ $\binom{v}{w}$ solves

$$\begin{aligned} v_{(-\infty,0)} &= v^0 \\ \dot{v}_{[0,\infty)} &= (Jv)_{[0,\infty)} \end{aligned}$$

and

$$\begin{aligned} w_{(-\infty,0)} &= w^0 \\ (N\dot{w})_{[0,\infty)} &= w_{[0,\infty)} \end{aligned}$$

ODE

$$v(t) = e^{Jt}v(0^-) \quad \forall t \geq 0$$

In particular, $v(0^+) = v(0^-)$

Nilpotent DAE

$$w(t) = 0 \quad \forall t > 0$$

In particular, $w(0^+) = 0$

Altogether we have

$$\begin{pmatrix} v(0^+) \\ w(0^+) \end{pmatrix} = \begin{pmatrix} v(0^-) \\ 0 \end{pmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} v(0^-) \\ w(0^-) \end{pmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}x(0^-)$$

hence

$$x(0^+) = T \begin{pmatrix} v(0^+) \\ w(0^+) \end{pmatrix} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}x(0^-) = \Pi_{(E,A)}x(0^-)$$

Existence of solution

Remarks

a) $\Pi_{(E,A)} = T \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] T^{-1}$ does not depend on the specific choice of T.

b) At this point we haven't actually shown that (ITP) has a solution!

Theorem

Let (E, A) be regular. In the correct distributional solution space the ITP has a unique solution for all f.

In particular, jump and Dirac impulses at $t = 0$ are uniquely determined.

Attention

Choosing the right solution space is crucial and not immediately clear!

Here: Solution space = piecewise-smooth distributions $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$

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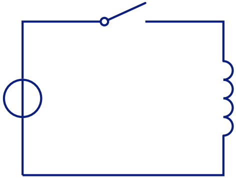
Definition



Switch → Different DAE models (=modes) depending on time-varying position of switch

Definition (Switched DAE)

Switching signal $\sigma : \mathbb{R} \to \{1, \ldots, N\}$ picks mode at each time $t \in \mathbb{R}$:

$$\begin{aligned} E_{\sigma(t)}\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) &= C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t) \end{aligned} \qquad \text{(swDAE)}$$

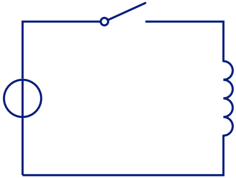
Attention

Each mode might have different consistency spaces

⇒ inconsistent initial values at each switch

⇒ distributional solutions, i.e. $x \in \mathbb{D}^n_{\text{pw}\mathcal{C}^\infty}$, $u \in \mathbb{D}^m_{\text{pw}\mathcal{C}^\infty}$, $y \in \mathbb{D}^p_{\text{pw}\mathcal{C}^\infty}$

Definition



Switch → Different DAE models (=modes) depending on time-varying position of switch

Definition (Switched DAE)

Switching signal $\sigma : \mathbb{R} \to \{1, \ldots, N\}$ picks mode at each time $t \in \mathbb{R}$:

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned} \tag{swDAE}$$

Attention

Each mode might have different consistency spaces
⇒ inconsistent initial values at each switch
⇒ distributional solutions, i.e. $x \in \mathbb{D}^n_{\text{pw}\mathcal{C}^\infty}$, $u \in \mathbb{D}^m_{\text{pw}\mathcal{C}^\infty}$, $y \in \mathbb{D}^p_{\text{pw}\mathcal{C}^\infty}$

Existence and uniqueness of solutions for (swDAE)

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned} \qquad \text{(swDAE)}$$

$$\Sigma_0 := \left\{ \sigma : \mathbb{R} \to \{1, \ldots, N\} \;\middle|\; \begin{array}{l} \sigma \text{ is piecewise constant and} \\ \sigma\big|_{(-\infty,0)} \text{ is constant} \end{array} \right\}.$$

Corollary (from previous section)

Consider (swDAE) *with regular* (E_p, A_p) $\forall p \in \{1, \ldots, N\}$. *Then*

$$\forall\; u \in \mathbb{D}^m_{\text{pw}\mathcal{C}^\infty} \;\forall\; \sigma \in \Sigma_0 \;\exists\; \textit{solution}\; x \in \mathbb{D}^n_{\text{pw}\mathcal{C}^\infty}$$

and $x(0-)$ *uniquely determines* x.

Sufficient conditions for impulse-freeness

Question

When are all solutions of homogenous (swDAE) $E_\sigma \dot{x} = A_\sigma x$ impulse free?

Note: Jumps are OK.

Lemma (Sufficient conditions)

- *(E_p, A_p) all have index one (i.e. $N_p = 0$ in QWF)*
  $\Rightarrow$ (swDAE) impulse free
- *all consistency spaces of (E_p, A_p) coincide (i.e. Wong limits $\mathcal{V}_p^*$ are identical)*
  $\Rightarrow$ (swDAE) impulse free

Scetch of proof

- Index-1-case: Consider nilpotent DAE-ITP:

$$
\begin{aligned}
&(N\dot{w})_{[0,\infty)} = w_{[0,\infty)} \\
&\Rightarrow 0 = w_{[0,\infty)} \\
&\Rightarrow w[0] := w_{[0,0]} = 0
\end{aligned}
$$

  Hence an inconsistent initial value does not induce Dirac-impulse

- Same consistency space for all modes

  $\Rightarrow$ no inconsistent initial values at switch

  $\Rightarrow$ no jumps and no Dirac-impulses

Characterization of impulse-freeness

Theorem (Impulse-freeness)

The switched DAE $E_\sigma \dot{x} = A_\sigma x$ is impulse free $\forall \sigma \in \Sigma_0$

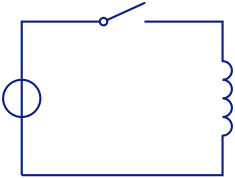
$$\Leftrightarrow \quad E_q(I - \Pi_q)\Pi_p = 0 \quad \forall p, q \in \{1, \ldots, N\}$$

where $\Pi_p := \Pi_{(E_p, A_p)}$, $p \in \{1, \ldots, N\}$ is the consistency projector.

Remark

- Index-1-case $\Rightarrow$ $E_q(I - \Pi_q) = 0 \ \forall q$
- Consistency spaces equal $\Rightarrow$ $(I - \Pi_q)\Pi_p = 0 \ \forall p, q$

Circuit example



$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 \\ L & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \qquad (E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ L & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right)$$

$$\Pi_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \Pi_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

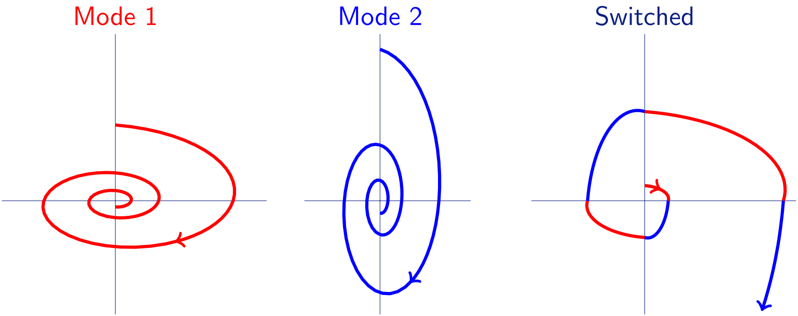
$$E_1(I - \Pi_1)\Pi_2 = \begin{bmatrix} 0 & 0 \\ L & 0 \end{bmatrix} \neq 0 \quad \Rightarrow \text{ impulses possible}$$

Stability

Question

All modes stable $\overset{?}{\Rightarrow}$ Switched system stable?

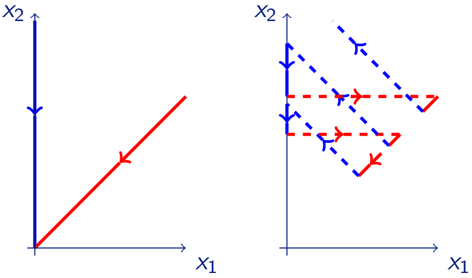
Answer: NO! Already false for switched ODEs:



Jumps and Stability: Example

$$E_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \; A_1 = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

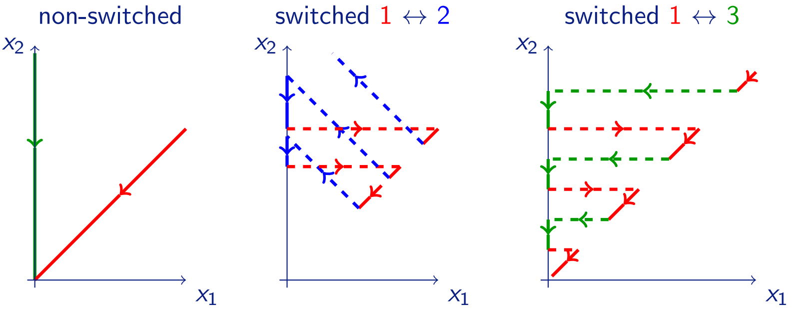
non-switched | switched $1 \leftrightarrow 2$



jumps destabilize

Jumps and Stability: Example

$$E_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ A_1 = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ A_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$



jumps destabilize jumps do not destabilize

Summary

1. Motivation: Modeling of electrical circuits
2. DAEs: Differences to ODEs
3. Special DAE-cases
   - Nilpotent DAEs
   - Underdetermined DAEs
   - Overdetermined DAEs
4. Equivalence and quasi-Kronecker form/quasi-Weierstrass form
5. Wong sequences
6. Inconsistent initial values
   - Motivating example
   - Consistency projector
7. Switched DAEs
   - Definition and solution theory
   - Impulse-freeness
   - Stability

Literature

- **S. Trenn (2013): Solution concepts for linear DAEs: a survey.**
  Chapter 4 in: A. Ilchmann, T. Reis (eds.), *Surveys in Differential-Algebraic Equations I*, Springer Verlag.
  doi:10.1007/978-3-642-34928-7_4
- **S. Trenn (2012): Switched differential algebraic equations**
  Chapter 6 in: F. Vasca, L. Iannelli (eds.), *Dynamics and Control of Switched Electronic Systems*, Springer Verlag.
  doi:10.1007/978-1-4471-2885-4_6
- **S. Trenn (2013): Stability of switched DAEs**
  Chapter 3 in: J. Daafouz, S. Tarbouriech, M. Sigalotti (eds.), *Hybrid Systems with Constraints*, Wiley.
  doi:10.1002/9781118639856.ch3

Preprints and slides available at research.stephantrenn.de