Stability of switched DAEs

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Switched DAEs

Switched linear DAE (differential algebraic equation)

(swDAE) $E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t)$ or short $E_\sigma \dot{x} = A_\sigma x$

with

- switching signal $\sigma : \mathbb{R} \to \{1, 2, \ldots, \mathtt{P}\}$
  - piecewise constant, right-continuous
  - locally finitely many jumps
- matrix pairs $(E_1, A_1), \ldots, (E_\mathtt{P}, A_\mathtt{P})$
  - $E_p, A_p \in \mathbb{R}^{n\times n}$, $p = 1, \ldots, \mathtt{P}$
  - (E_p, A_p) regular, i.e. $\det(E_p s - A_p) \not\equiv 0$

Motivation and questions

Why switched DAEs $E_\sigma \dot{x} = A_\sigma x$?

1. modeling of electrical circuits with switches
2. DAEs $E\dot{x} = Ax + Bu$ with switched feedback

$$u(t) = F_{\sigma(t)}x(t) \quad \text{or}$$
$$u(t) = F_{\sigma(t)}x(t) + G_{\sigma(t)}\dot{x}(t)$$

3. approximation of time-varying DAEs $E(t)\dot{x} = A(t)x$ via piecewise-constant DAEs

Question

$E_p\dot{x} = A_p x$ asymp. stable $\forall p \quad \overset{?}{\Rightarrow} \quad E_\sigma \dot{x} = A_\sigma x$ asymp. stable $\forall \sigma$

Example 1: jumps and stability

Example 1a:

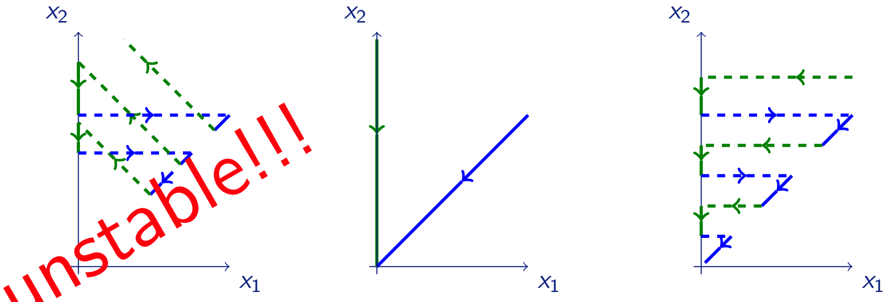
$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \right)$$

$$(E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

Example 1b:

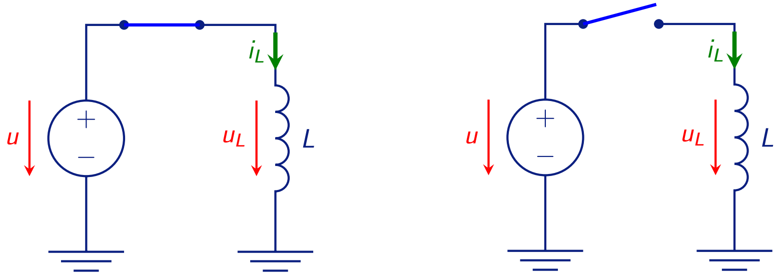
$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \right)$$

$$(E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$



Remark: $V(x) = x_1^2 + x_2^2$ is Lyapunov function for all subsystem

Example 2: impulses in solutions

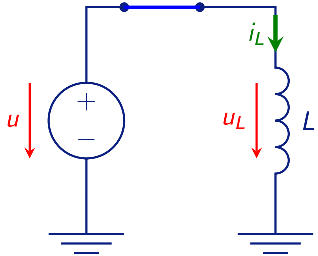


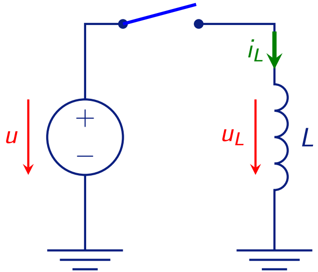
constant input: $\dot{u} = 0$

inductivity law: $L \frac{\mathrm{d}}{\mathrm{d}t} i_L = u_L$

switch dependent: $0 = u_L - u$ $\quad$ $0 = i_L$

Example 2: impulses in solutions





$$x = [u, i_L, u_L]^\top$$

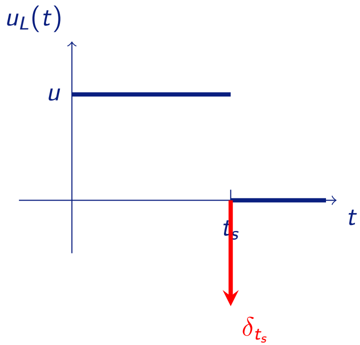
$$\begin{bmatrix}1 & 0 & 0\\ 0 & L & 0\\ 0 & 0 & 0\end{bmatrix}\dot{x} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ -1 & 0 & 1\end{bmatrix}x \qquad \begin{bmatrix}1 & 0 & 0\\ 0 & L & 0\\ 0 & 0 & 0\end{bmatrix}\dot{x} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}x$$

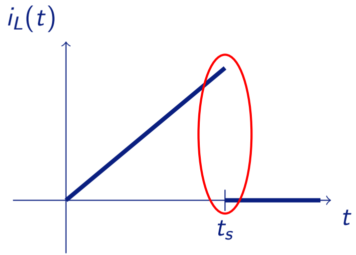
Solution of example

$L\frac{\mathrm{d}}{\mathrm{d}t}i_L = u_L, \quad 0 = u_L - u$ or $0 = i_L$

u constant, $i_L(0) = 0$

switch at $t_s > 0$: $\sigma(t) = \begin{cases} 1, & t < t_s \\ 2, & t \geq t_s \end{cases}$





Observations

Solutions

- modes have constrained dynamics: consistency spaces
- switches $\Rightarrow$ inconsistent initial values
- inconsistent initial values $\Rightarrow$ jumps in x

Stability

- common Lyapunov function not sufficient
- stability depends on jumps

Impulses

- switching $\Rightarrow$ Dirac impulse in solution x
- Dirac impulse = infinite peak $\Rightarrow$ instability

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Solutions for unswitched DAEs

Consider $E\dot{x} = Ax$.

Theorem (Weierstrass 1868)

(E, A) *regular* $\Leftrightarrow$
$\exists S, T \in \mathbb{R}^{n\times n}$ *invertible:*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right),$$

N *nilpotent,* $T = [V, W]$

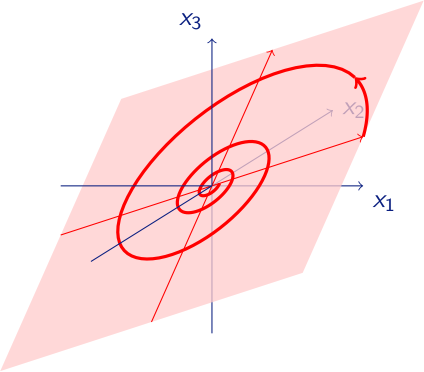
Corollary (for regular (E, A))

x *solves* $E\dot{x} = Ax \Leftrightarrow$

$$x(t) = Ve^{Jt}v_0$$

$V \in \mathbb{R}^{n\times n_1}$, $J \in \mathbb{R}^{n_1\times n_1}$, $v_0 \in \mathbb{R}^{n_1}$.
Consistency space: $\mathfrak{C}_{(E,A)} := \operatorname{im} V$

$$(E, A) = \left(\begin{bmatrix} 0 & 4 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -4\pi & -4 & 0 \\ -1 & 4\pi & 0 \\ -1 & -4 & 4 \end{bmatrix} \right)$$



$$V = \begin{bmatrix} 0 & 4 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \; J = \begin{bmatrix} -1 & -4\pi \\ \pi & -1 \end{bmatrix}$$

Consistency projector

Observation

$$\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \begin{pmatrix} \dot v \\ \dot w \end{pmatrix} = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \begin{pmatrix} v \\ w \end{pmatrix}$$

Consistent initial value: $\begin{pmatrix} v_0 \\ 0 \end{pmatrix}$, because $N\dot w = w \;\Leftrightarrow\; w \equiv 0$

arbitrary initial value $\begin{pmatrix} v_0 \\ w_0 \end{pmatrix} \overset{\Pi}{\mapsto} \begin{pmatrix} v_0 \\ 0 \end{pmatrix}$ consistent initial value

Definition (Consistency projector for regular (E, A))

Let $S, T \in \mathbb{R}^{n\times n}$ be invertible with $(SET, SAT) = \left(\left[\begin{smallmatrix} I & 0 \\ 0 & N \end{smallmatrix}\right], \left[\begin{smallmatrix} J & 0 \\ 0 & I \end{smallmatrix}\right] \right)$:

$$\Pi_{(E,A)} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$$

Remark: $\Pi_{(E,A)}$ can be calculated easily and directly from (E, A) (via the Wong sequences)

Lyapunov functions for regular (E, A)

Definition (Lyapunov function for $E\dot{x} = Ax$)

$Q = \overline{Q}^\top > 0$ on $\mathfrak{C}_{(E,A)}$ and $P = \overline{P}^\top > 0$ solutions of

$$A^\top PE + E^\top PA = -Q \quad \text{(generalize Lyapunov equation)}$$

Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0} : x \mapsto (Ex)^\top PEx$

V monotonically decreasing along solutions:

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} V\big(x(t)\big) &= \big(Ex(t)\big)^\top PE\dot{x}(t) + \big(E\dot{x}(t)\big)^\top PEx \\ &= x(t)^\top E^\top PAx(t) + x(t)^\top A^\top PEx(t) \\ &= -x(t)^\top Qx(t) < 0 \end{aligned}$$

Theorem (Owens & Debeljkovic 1985)

$E\dot{x} = Ax$ asymptotically stable $\quad\Leftrightarrow\quad \exists$ Lyapunov function

Contents

Distribution theory - basics

Distributions - overview

- generalized functions
- arbitrarily often differentiable
- Dirac impulse δ_0 is "derivative" of unit jump $\mathbb{1}_{[0,\infty)}$

Two different formal approaches

1. functional analytical: dual of the space test functions (L. Schwartz 1950)
2. axiomatic: space of all "derivatives" of continuous functions (J. Sebastião e Silva 1954)

Dilemma

(swDAE) $E_\sigma \dot{x} = A_\sigma x$

Problem

Multiplication of non smooth coefficients E_σ, A_σ with general distribution x not defined!

switched DAEs

- example: distributional solutions
- multiplication with non-smooth coefficients

distributions

- multiplication with non-smooth coefficients not well-defined
- *initial value problems cannot be formulated*

Underlying problem

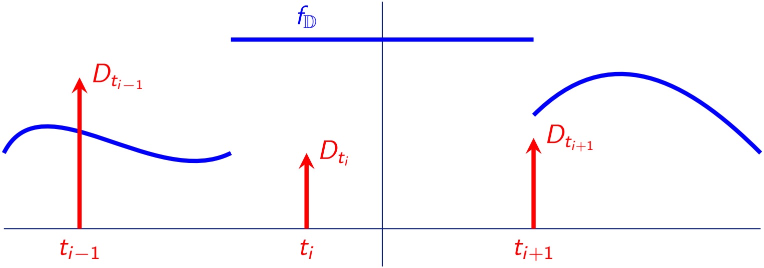
Space of distributions too big.

Piecewise-smooth distributions

define a more suitable, smaller space:

Definition (Piecewise-smooth distributions $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$)

$$\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty} := \left\{ f_{\mathbb{D}} + \sum_{t \in T} D_t \;\middle|\; \begin{array}{l} f \in \mathcal{C}^\infty_{\mathrm{pw}}, \\ T \subseteq \mathbb{R} \text{ locally finite}, \\ \forall t \in T : D_t = \sum_{i=0}^{n_t} a_i^t \delta_t^{(i)} \end{array} \right\}$$



Properties of $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$

- multiplication with $\mathcal{C}^\infty_{\mathrm{pw}}$-functions well defined (Fuchssteiner multiplication)
- left and right evaluation at $t \in \mathbb{R}$ possible: $D(t-), D(t+)$
- impulse at $t \in \mathbb{R}$: $D[t]$

(swDAE) $\qquad E_\sigma \dot{x} = A_\sigma x$

Application to (swDAE)

x solves **(swDAE)** $:\Leftrightarrow$ $x \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ and **(swDAE)** holds in $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$

Theorem (Existence and uniqueness of solutions, T. 2009)

(E_p, A_p) *regular* $\forall p$ $\Leftrightarrow$ **(swDAE)** *uniquelly solvable* $\forall \sigma\ \forall x(0) \in \mathbb{R}^n$

Intermediate summary: problems and its solutions

(swDAE) $E_\sigma \dot{x} = A_\sigma x$

1. stability criteria for single DAEs $E_p \dot{x} = A_p x$
   $\Rightarrow$ Lyapunov functions
2. **no classical solutions**
   $\Rightarrow$ allow jumps in solutions
3. How does inconsistent initial value jump to consistent one?
   $\Rightarrow$ Consistency projectors $\Pi_{(E_1,A_1)}, \ldots, \Pi_{(E_N,A_N)}$
4. differentiation of jumps
   $\Rightarrow$ space of distributions as solution space
5. **multiplication with non-smooth coefficients**
   $\Rightarrow$ space of piecewise-smooth distributions
   $\Rightarrow$ existence and uniqueness of solutions

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Asymptotic stability and impulse free solutions

Definition (Asymptotic stability of switched DAE)

(swDAE) asymptotically stable
$:\Leftrightarrow$ x is impulse free* and $x(t\pm) \to 0$ for $t \to \infty$

* i.e. $x[t] = 0\ \forall t \in \mathbb{R}$; however jumps in x are still allowed

Let $\Pi_p := \Pi_{(E_p,A_p)}$ be the consistency projector of (E_p, A_p)

Impulse freeness condition

(IFC): $\forall p, q \in \{1, \ldots, N\}: \quad E_q(I - \Pi_q)\Pi_p = 0$

Theorem (T. 2009)

(IFC) $\quad\Leftrightarrow\quad$ *all solutions of $E_\sigma \dot{x} = A_\sigma x$ are impulse free $\forall\sigma$*

Stability for arbitrary switching

Consider **(swDAE)** with:

($\exists$V$_\mathbf{p}$): $\forall p \in \{1, \ldots, \mathtt{P}\}$ $\exists$ Lyapunov function V_p for (E_p, A_p)

i.e. each DAE $E_p\dot{x} = A_p x$ is asymptotically stable

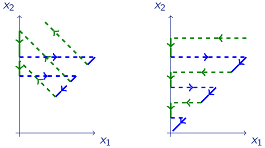
Lyapunov jump condition

(LJC): $\forall p, q = 1, \ldots, N$ $\forall x \in \mathfrak{C}_{(E_p, A_p)}$: $\quad V_q(\Pi_q x) \leq V_p(x)$

Theorem (Liberzon & T. 2009)

(IFC) $\wedge$ **($\exists$V$_\mathbf{p}$)** $\wedge$ **(LJC)** $\Rightarrow$ **(swDAE)** *asymtotically stable* $\forall\sigma$

Examples 1a and 1b fulfill **(IFC)** and **($\exists$V$_\mathbf{p}$)**, but only 1b fulfills **(LJC)**



Slow switching

Consider the set of switching signals with dwell time $\tau > 0$:

$$\Sigma^\tau := \left\{ \sigma : \mathbb{R} \to \{1, \ldots, N\} \;\middle|\; \begin{array}{l} \forall \text{ switching times} \\ t_i \in \mathbb{R}, i \in \mathbb{Z} : \\ t_{i+1} - t_i \geq \tau \end{array} \right\}.$$

Theorem (Liberzon & T. 2009)

$\exists \tau > 0$: **(IFC)** $\wedge$ **($\exists \mathbf{V_p}$)** $\Rightarrow$ **(swDAE)** *asymptotically stable* $\forall \sigma \in \Sigma^\tau$

Reminder:
(IFC): $\forall p, q \in \{1, \ldots, N\} : \quad E_q(I - \Pi_q)\Pi_p = 0$

Examples 1a and 1b both fulfill **(IFC)** and **($\exists \mathbf{V_p}$)**
$\Rightarrow$ both examples are asymptotically stable for slow switching

Generalization to nonlinear switched DAEs

Previous results can be generalized to nonlinear switched DAEs:

$$E_\sigma(x)\dot{x} = f_\sigma(x)$$

Then **(IFC)** has to be replaced by

$$\forall p, q \in \{1, \ldots, \mathrm{P}\}\ \forall x_0^- \in \mathfrak{C}_p\ \exists \text{ unique } x_0^+ \in \mathfrak{C}_q :\ \ x_0^+ - x_0^- \in \ker E_q(x_0^+)$$

where $\mathfrak{C}_p$ is the consistency manifold of $E_p(x)\dot{x} = f_p(x)$

See our Automatica paper "Switched nonlinear differential algebraic equations: Solution theory, Lyapunov functions, and stability" (2012)

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Commutativity and stability of switched ODEs

Theorem (Narendra and Balakrishnan 1994)

Consider switched ODE

(swODE) $\dot{x} = A_\sigma x$

with A_p Hurwitz, $p \in \{1, 2, \ldots, \mathtt{P}\}$ and commuting A_p, i.e.

$$[A_p, A_q] := A_p A_q - A_q A_p = 0 \quad \forall p, q \in \{1, 2, \ldots, \mathtt{P}\} \qquad \text{(C)}$$

$\Rightarrow$ **(swODE)** *asymptotically stable $\forall\sigma$.*

Proof idea: Consider switching times $t_0 < t_1 < \ldots < t_k < t$ and $p_i := \sigma(t_i+)$, then

$$\begin{aligned} x(t) &= e^{A_{p_k}(t-t_k)} e^{A_{p_{k-1}}(t_k - t_{k-1})} \cdots e^{A_{p_1}(t_2-t_1)} e^{A_{p_0}(t_1-t_0)} x_0 \\ &\overset{\text{(C)}}{=} e^{A_1 \Delta t_1} e^{A_2 \Delta t_2} \cdots e^{A_{\mathtt{P}} \Delta t_{\mathtt{P}}} x_0 \end{aligned}$$

and $\Delta t_p \to \infty$ for at least one p and $t \to \infty$.

Generalization to **(swDAE)**

(swDAE) $E_\sigma \dot{x} = A_\sigma x$

Generalization - Questions

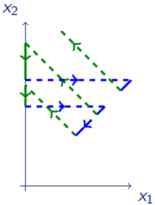
- Which matrices have to commute?
- What about the jumps?

Example 1a:

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \right)$$

$$(E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

$[A_1, A_2] = 0$, but unstable for fast switching



The matrix A^{diff}

Let (E,A) regular with $(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right)$, N nilpotent

consistency projector: $\Pi_{(E,A)} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$

Definition (differential "projector")

$$\Pi^{\text{diff}}_{(E,A)} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S$$

Lemma (Dynamics of DAE, Tanwani & T. 2010)

x *solves* $E\dot{x} = Ax \quad \Rightarrow \quad \dot{x} = \underbrace{\Pi^{\text{diff}}_{(E,A)} A}_{=:A^{\text{diff}}} x$

Note: $A^{\text{diff}} = T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$, hence $[A^{\text{diff}}, \Pi_{(E,A)}] = 0$

Commutativity condition

(swDAE) $E_\sigma \dot{x} = A_\sigma x$

Theorem (Liberzon, T., Wirth 2011)

(IFC) $\wedge$ **($\exists$V$_p$)** $\wedge$

$$[A_p^{\text{diff}}, A_q^{\text{diff}}] = 0 \quad \forall p, q \in \{1, 2, \ldots, \mathtt{P}\} \qquad \text{(C)}$$

$\Rightarrow$ **(swDAE)** *is asymptotically stable* $\forall \sigma$.

(IFC) $\wedge$ **($\exists$V$_p$)** $\wedge$ (C) $\Rightarrow \exists$ *common quadratic Lyapunov function with*

$$V(\Pi_p x) \leq V(x) \quad \forall x \; \forall p$$

Remarkable: No explicit condition on jumps!

Proof idea

Proof idea:

$$[A_p^{\text{diff}}, A_q^{\text{diff}}] = 0 \quad \forall p, q \in \{1, 2, \dots, \mathtt{P}\} \tag{C}$$

implies

$$[\Pi_p, A_q^{\text{diff}}] = 0 \quad \wedge \quad [\Pi_p, \Pi_q] = 0.$$

Consider switching times $t_0 < t_1 < \ldots < t_k < t$ and $p_i := \sigma(t_i+)$, then

$$\begin{aligned} x(t) &= e^{A_{p_k}^{\text{diff}}(t-t_k)} \Pi_{p_k} e^{A_{p_{k-1}}^{\text{diff}}(t_k - t_{k-1})} \Pi_{p_{k-1}} \cdots e^{A_{p_1}^{\text{diff}}(t_2-t_1)} \Pi_{p_1} e^{A_{p_0}^{\text{diff}}(t_1-t_0)} \Pi_{p_0} x_0 \\ &\overset{(C)}{=} e^{A_1^{\text{diff}} \Delta t_1} \Pi_1 \, e^{A_2^{\text{diff}} \Delta t_2} \Pi_2 \cdots e^{A_{\mathtt{P}}^{\text{diff}} \Delta t_{\mathtt{P}}} \Pi_{\mathtt{P}} x_0 \end{aligned}$$

and $\Delta t_p \to \infty$ for at least one p and $t \to \infty$.

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Evolution operator

$$x(t) = \underbrace{e^{A_k^{\text{diff}}(t-t_k)}\Pi_k e^{A_{k-1}^{\text{diff}}(t_k-t_{k-1})}\Pi_{k-1}\cdots e^{A_1^{\text{diff}}(t_2-t_1)}\Pi_1 e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0}_{=:\,\Phi^\sigma(t,t_0)} x(t_0-)$$

Let $\mathcal{M} := \{\ (A_p^{\text{diff}}, \Pi_p)\ \mid \text{corresponding to } (E_p, A_p), p = 1, \ldots, \mathtt{p}\ \}$.

Definition (Set of all evolution matrices with fixed time span $t > 0$)

$$\begin{aligned}\mathcal{S}_t &:= \{\ \Phi^\sigma(t,0)\ \mid \sigma \text{ arbitrary switching signal }\ \}\\ &= \left\{ \prod_{i=0}^{k} e^{A_i^{\text{diff}}\tau_i}\Pi_i \ \middle|\ (A_i^{\text{diff}}, \Pi_i) \in \mathcal{M},\ \sum_{i=0}^{k}\tau_i = \Delta t,\ \tau_i > 0 \right\}\end{aligned}$$

Lemma (Semi group, T. & Wirth 2012)

The set $\mathcal{S} := \bigcup_{t>0}\mathcal{S}_t$ is a semi group with

$$\mathcal{S}_{s+t} = \mathcal{S}_s\mathcal{S}_t := \{\ \Phi_s\Phi_t\ \mid \Phi_s \in \mathcal{S}_s, \Phi_t \in \mathcal{S}_t\ \}$$

Exponential growth bound

Definition (Exponential growth bound)

For $t > 0$ the *exponential growth bound* of $E_\sigma \dot{x} = A_\sigma x$ is

$$\lambda_t(\mathcal{S}_t) := \sup_{\Phi_t \in \mathcal{S}_t} \frac{\ln \|\Phi_t\|}{t} \in \mathbb{R} \cup \{-\infty, \infty\}$$

Definition implies for all solutions x of $E_\sigma \dot{x} = A_\sigma x$:

$$\|x(t)\| = \|\Phi_t x(0-)\| \leq \|\Phi_t\| \, \|x(0-)\| \leq e^{\lambda_t(\mathcal{S}_t)\, t} \|x(0-)\|$$

Difference to switched ODEs without jumps

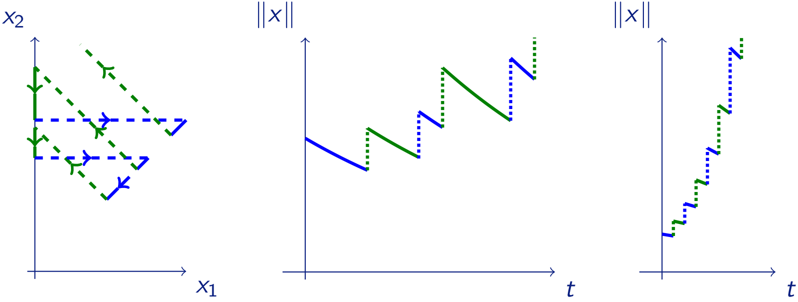
$\lambda_t(\mathcal{S}_t) = \pm\infty$ is possible!

All jumps are trivial, i.e. $\Pi_p = 0 \quad \Rightarrow \quad \lambda_t(\mathcal{S}_t) = -\infty$

Infinite exponential growth bound

Example 1a revisited:

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \right) \quad (E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$



For small dwell times: $\Phi_t \approx (\Pi_1 \Pi_2)^k = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^k = 2^{k-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Lyapunov exponent of a switched DAE

Theorem (Boundedness of $\mathcal{S}_t$, T. & Wirth 2012)

$\mathcal{S}_t$ is bounded $\Leftrightarrow$ the set of consistency projectors is product bounded

(swDAE) $E_\sigma \dot{x} = A_\sigma x$

Theorem (Lyapunov exponent well defined, T. & Wirth 2012)

Let the consistency projectors be product bounded and not all be trivial, then the (upper) Lyapunov exponent

$$\lambda(\mathcal{S}) := \lim_{t\to\infty} \lambda_t(\mathcal{S}_t) = \lim_{t\to\infty} \sup_{\Phi_t \in \mathcal{S}_t} \frac{\ln \|\Phi_t\|}{t}$$

of **(swDAE)** *is well defined and finite.*

Note that: **(swDAE)** uniformly exponentially stable

$:\Leftrightarrow \quad \exists M \geq 1, \mu > 0 : \|x(t)\| \leq M e^{-\mu t} \|x(0-)\| \quad \forall t \geq 0$

$\Rightarrow \quad \lambda(\mathcal{S}) \leq -\mu < 0$

Converse Lyapunov theorem for switched DAEs

For $\varepsilon > 0$ define "Lyapunov norm"

$$|||x|||_\varepsilon := \sup_{t>0} \sup_{\Phi_t \in \mathcal{S}_t} e^{-(\lambda(\mathcal{S})+\varepsilon)t} \|\Phi_t x\|$$

(swDAE) $E_\sigma \dot{x} = A_\sigma x$

Theorem (Converse Lyapunov theorem, T. & Wirth 2012)

(swDAE) *is uniformly exponentially stable* $\forall \sigma$
$\Rightarrow$ $V = |||\cdot|||_\varepsilon$ *is Lyapunov function for sufficiently small* $\varepsilon > 0$

In particular: $V(\Pi x) \leq V(x)$ for all consistency projectors Π

Non-smooth Lyapunov function

$|||\cdot|||_\varepsilon$ in general non-smooth. Smoothification as in Yin, Sontag & Wang 1996 might violate jump condition!

Summary

(swDAE) $E_\sigma \dot{x} = A_\sigma x$

- solution theory
  - no classical solutions: jumps and impulses
  - impulse freeness condition **(IFC)**
  - jumps are still allowed
- stability conditions
  - multiple Lyapunov functions with jump condition **(LJC)**
  - slow switching
  - commutativity (quadratic Lyapunov function)
  - converse Lyapunov theorem

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