On averaging for switched linear differential algebraic equations

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12th European Control Conference (ECC'13)
Thursday, 18.07.2013, ThB5.3, 14:20

TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

Contents

Averaging: Basic idea

switched system	$\approx$ fast switching	non-switched average system

Application

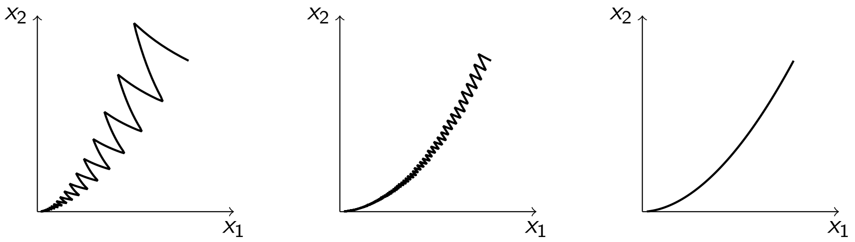
- Fast switches occurs at
  - Pulse width modulation
  - „Sliding mode"-control
  - In general: fast digital controller
- Simplified analyses
  - Stability for sufficiently fast switching
  - In general: desired (approximate) behavior via suitable switching

Simple example

Example

$$\dot{x} = A_\sigma x, \quad A_1 = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \quad \sigma : \mathbb{R} \to \{1, 2\} \text{ periodic}$$

switching frequency $\longrightarrow$ ∞



Fixed duty cycle for varying switching frequency (here 45 : 5555 : 45)

Simple example

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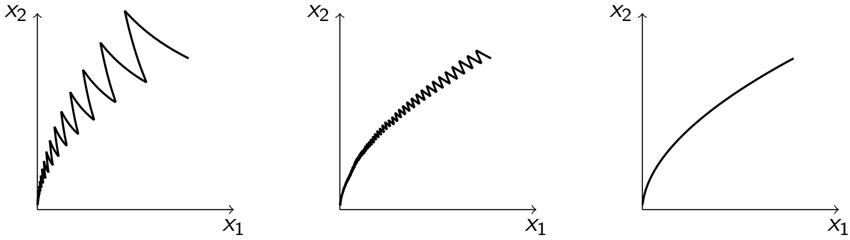


Fixed duty cycle for varying switching frequency (here 45 : 5555 : 45)

Averaging result for switched linear ODEs

Switched linear ODE

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad x(0) = x_0$$

where $\sigma : \mathbb{R} \to \{1, 2\}$ is periodic with periode $p > 0$ and duty cycle $d > 0$

Theorem (Brockett & Wood 1974)

Let the average system be given by

$$\dot{x}_{\text{av}} = A_{\text{av}}x_{\text{av}}, \quad x_{\text{av}}(0) = x_0$$

and

$$A_{\text{av}} := dA_1 + (1 - d)A_2.$$

Then on every compact time interval:

$$\|x(t) - x_{\text{av}}(t)\| = O(p).$$

Switched DAEs

Modeling of electrical circuits with switches yields

Switched differential-algebraic equations (DAEs)

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) \qquad \textbf{(swDAE)}$$

Question

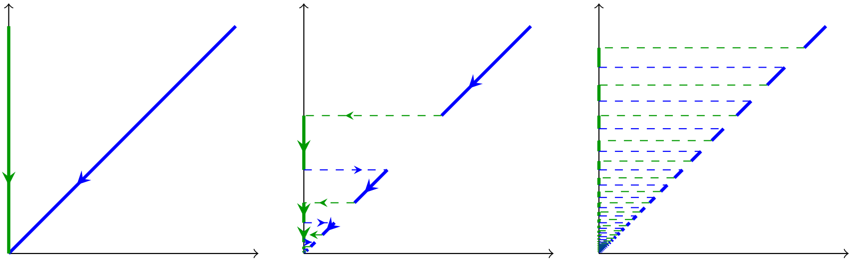
Does a similar result also hold for switched DAEs?

A counterexample

Consider $E_\sigma \dot{x} = A_\sigma x$ with

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \right), \quad (E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

no switching slow switching fast switching



System class

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) \qquad \textbf{(swDAE)}$$

Assumptions

- $\sigma : [0,\infty) \to \{1,2\}$ periodic with periode $p > 0$
- $d \in (0,1)$ is duty cycle for mode 1
- matrix pairs (E_k, A_k), $k \in \{1,2\}$, regular, i.e. $\det(sE_k - A_k) \not\equiv 0$

Theorem (Quasi-Weierstrass-form, Weierstrass 1868)

(E, A) regular $\quad\Leftrightarrow\quad \exists T, S$ invertible:

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \quad N \text{ nilpotent}$$

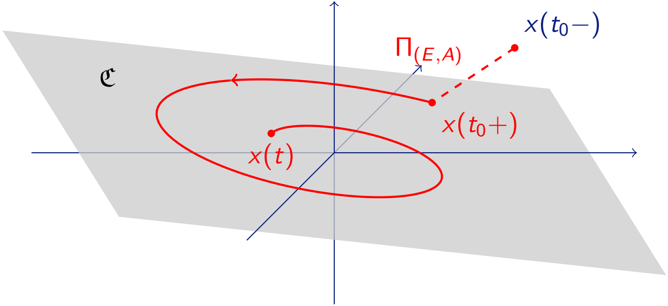
Note: S and T can easily be calculated via the Wong sequences

Non-switched DAEs: Properties

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \quad N \text{ nilpotent}$$

Definition (Consistency projector)

$$\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$$



Solution characterization of DAEs

$$(SET, SAT) = \left(\left[\begin{smallmatrix} I & 0 \\ 0 & N \end{smallmatrix}\right], \left[\begin{smallmatrix} J & 0 \\ 0 & I \end{smallmatrix}\right] \right), \quad N \text{ nilpotent}$$

Definition (Flow matrix A^{diff})

$$A^{\text{diff}} := T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$$

Theorem (Solution characterization, Tanwani & T. 2010)

Consider DAE $E\dot{x} = Ax$ with regular matrix pair (E, A) and corresponding consistency projector $\Pi_{(E,A)}$ and A^{diff}

$\Rightarrow$

$$x(t) = e^{A^{\text{diff}}(t-t_0)} \Pi_{(E,A)} x(t_0-) \in \mathfrak{C} \quad t \in (t_0, \infty).$$

Remark: At t_0 the presence of Dirac-impulses is possible!

Solution behavior for switched DAEs

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) \qquad \textbf{(swDAE)}$$

with consistency projectors Π_k and flow matrices A_k^{diff}

Theorem (Impulse freeness, T. 2009)

All solutions of (**swDAE**) *are impulse free, if*

$$E_1(I-\Pi_1)\Pi_2 = 0 = E_2(I-\Pi_2)\Pi_1, \qquad \textbf{(IFC)}$$

Corollary

All solutions of (**swDAE**) *satisfying* (**IFC**) *are given by*

$$\begin{aligned} x(t) &= e^{A_{p_{2i}}^{\text{diff}}(t-t_{2i})}\Pi_{p_{2i}} \cdots e^{A_2^{\text{diff}}(t_3-t_2)}\Pi_2 e^{A_1^{\text{diff}}(t_2-t_1)}\Pi_1 x(t_1-) \\ &= M(t-t_i)\left(e^{A_2^{\text{diff}}(1-d)}\Pi_2 e^{A_1^{\text{diff}}d}\Pi_1\right)^i x(t_1-) \end{aligned}$$

Contents

Condition on consistency projectors

Assumption: commutative projectors

$$\Pi_1\Pi_2 = \Pi_2\Pi_1 \tag{C}$$

Lemma

(C) $\Rightarrow$ $\operatorname{im}\Pi_1\Pi_2 = \operatorname{im}\Pi_1 \cap \operatorname{im}\Pi_2$

Remark: $\operatorname{im}\Pi_1 \cap \operatorname{im}\Pi_2 = \mathfrak{C}_1 \cap \mathfrak{C}_2$ and obviously the average system, if it exists, can only have solutions within the intersection of the consistency spaces, hence the projector

$$\Pi_\cap := \Pi_1\Pi_2$$

plays a crucial role!

In the example it was: $\Pi_1\Pi_2 = \Pi_1 \neq \Pi_2 = \Pi_2\Pi_1$

Main result

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) \qquad \textbf{(swDAE)}$$

$$\Pi_1\Pi_2 = \Pi_2\Pi_1 \qquad \textbf{(C)}$$

Theorem (Averaging for switched DAEs)

Consider impulse free (**swDAE**) *with consistency projectors* Π_1, Π_2 *satisfying* (**C**) *and flow matrices* $A_1^{\text{diff}}, A_2^{\text{diff}}$. *The average system is*

$$\boxed{\dot{x}_{\text{av}} = \Pi_\cap A_{\text{av}}^{\text{diff}} \Pi_\cap x_{\text{av}}, \quad x_{\text{av}}(0) = \Pi_\cap x(0-)}$$

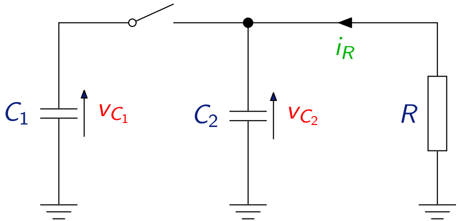
where $\Pi_\cap = \Pi_1\Pi_2$ *and*

$$A_{\text{av}}^{\text{diff}} := dA_1^{\text{diff}} + (1-d)A_2^{\text{diff}}.$$

Then $\forall t \in (0, T]$

$$\|x(t) - x_{\text{av}}(t)\| = O(p)$$

Example



Switch independent: $0 = v_{C_2} - R i_R$

Switch dependent:

open	closed
$C_1 \dot v_{C_1} = 0,$	$C_1 \dot v_{C_1} + C_2 \dot v_{C_2} = -i_R,$
$C_2 \dot v_{C_2} = -i_R,$	$0 = v_{C_1} - v_{C_2},$

$\Rightarrow$ switched DAE $E_\sigma \dot x = A_\sigma x$ with $x = (v_{C_1}, v_{C_2}, i_R)^\top$ given by

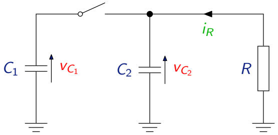
$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 & 0 \\ C_1 & 0 & 0 \\ 0 & C_2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & -R \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right)$$

$$(E_2, A_2) = \left(\begin{bmatrix} 0 & 0 & 0 \\ C_1 & C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -R \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \right)$$

Example

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 & 0 \\ C_1 & 0 & 0 \\ 0 & C_2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & -R \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right)$$

$$(E_2, A_2) = \left(\begin{bmatrix} 0 & 0 & 0 \\ C_1 & C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -R \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \right)$$



$\Rightarrow$ consistency projectors

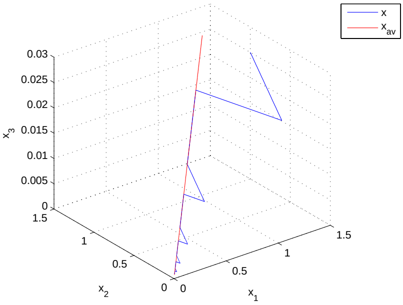
$$\Pi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{R} & 0 \end{bmatrix}, \quad \Pi_2 = \frac{1}{C_1+C_2} \begin{bmatrix} C_1 & C_2 & 0 \\ C_1 & C_2 & 0 \\ \frac{C_1}{R} & \frac{C_2}{R} & 0 \end{bmatrix}.$$

and (**C**) holds:

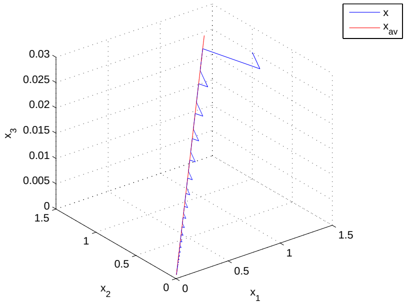
$$\Pi_1 \Pi_2 = \Pi_2 = \Pi_2 \Pi_1$$

Simulation results

$d_1 = 0.4, \quad p = 0.1$



$d_1 = 0.4, \quad p = 0.02$



Summary

- Generalization of classical averaging result to switched DAEs
  - average system does not exist in all cases
  - Additional condition for consistency projectors necessary
  - classical average matrix must be projected to the right space
- Further results and open questions
  - More than two modes (submitted to CDC'13)
  - Commutativity of consistency projectors necessary?
  - Impulses: Convergence in the sense of distributions?