Averaging for switched DAEs

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84th Annual Meeting of the GAMM 2013, Novi Sad
21.03.2013, Session S20.3, 14:50

Contents

Averaging: Basic idea

switched system	$\approx$ fast switching	non-switched average system

Application

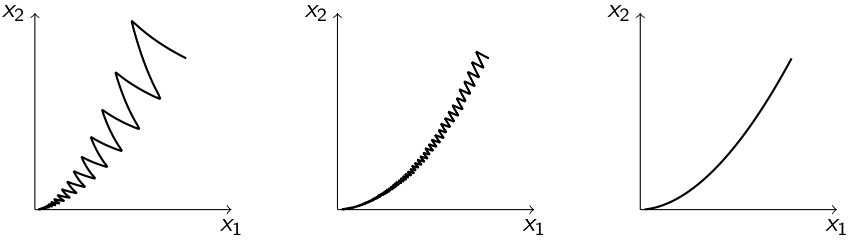
- Fast switches occurs at
  - Pulse width modulation
  - „Sliding mode"-control
  - In general: fast digital controller
- Simplified analyses
  - Stability for sufficiently fast switching
  - In general: desired behavior (approximate) via suitable switching

Simple example

Example

$$\dot{x} = A_\sigma x, \quad A_1 = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \quad \sigma : \mathbb{R} \to \{1,2\} \text{ periodic}$$

switching frequency $\longrightarrow$ ∞



Fixed duty cycle for varying switching frequency (here 45 : 5555 : 45)

Simple example

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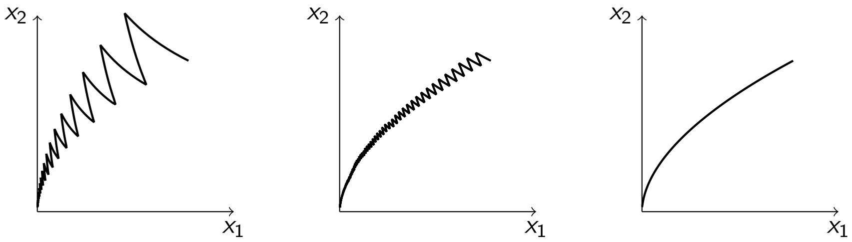


Fixed duty cycle for varying switching frequency (here 45 : 5555 : 45)

Averaging result for switched linear ODEs

Consider switched linear ODE

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad x(0) = x_0$$

with periodic $\sigma : \mathbb{R} \to \{1, 2, \ldots, M\}$ and period $p > 0$ and let $d_1, d_2, \ldots, d_M \geq 0$ with $d_1 + d_2 + \ldots + d_M = 1$ be the duty cycles of the switched system.

Theorem (Brockett & Wood 1974)

Let the averaged system be given by

$$\dot{x}_{\text{av}} = A_{\text{av}}x_{\text{av}}, \quad x_{\text{av}}(0) = x_0$$

and

$$A_{\text{av}} := d_1 A_1 + d_2 A_2 + \ldots + d_M A_M.$$

Then on every compact time interval:

$$\|x(t) - x_{\text{av}}(t)\| = O(p).$$

Switched DAEs

Modeling of electrical circuits with switches yields

Switched differential-algebraic equations (DAEs)

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) \qquad \textbf{(swDAE)}$$

Question

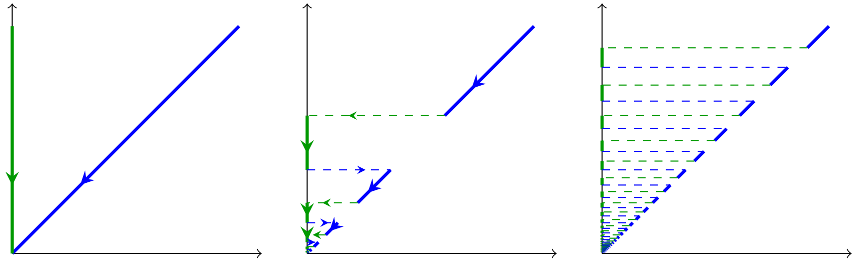
Does a similar result also hold for switched DAEs?

A counterexample

Consider $E_\sigma \dot{x} = A_\sigma x$ with

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \right), \quad (E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

no switching slow switching fast switching

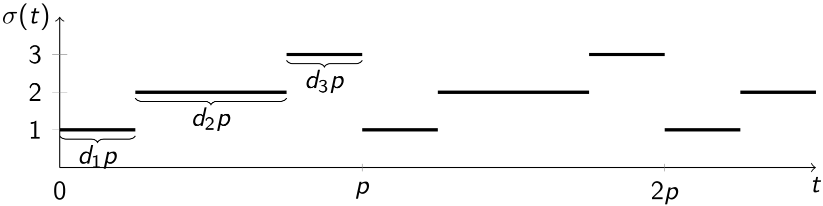


System class

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) \qquad \textbf{(swDAE)}$$

Assumptions

- $\sigma : [0,\infty) \to \{1,2,\ldots,M\}$ periodic with periode $p > 0$
- W.l.o.g.: σ monotonically increasing on $[0,p)$ and $d_k \in (0,1)$ is duty cycle for mode $k \in \{1,2,\ldots,M\}$
- matrix pairs (E_k, A_k), $k \in \{1,2,\ldots,M\}$, regular, i.e. $\det(sE_k - A_k) \not\equiv 0$



Non-switched DAEs: Properties

Theorem (Quasi-Weierstrass-form, Weierstrass 1868)

(E, A) *regular* $\Leftrightarrow$ $\exists T, S$ *invertible:*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \quad N \text{ nilpotent}$$

Definition (Consistency projector)

$$\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$$

Definition (Differential projector and A^{diff})

$$\Pi^{\text{diff}}_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S, \quad A^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)} A$$

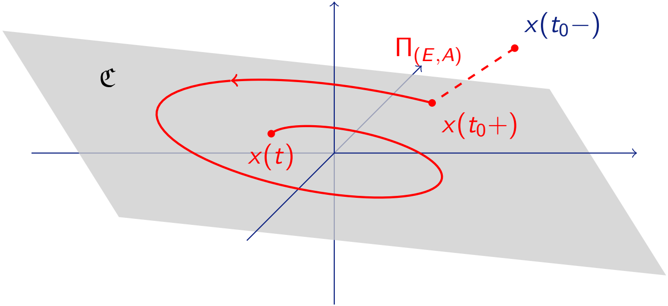
Solution characterization of DAEs

Theorem (Solution characterization, Tanwani & T. 2010)

Consider DAE $E\dot{x} = Ax$ with regular matrix pair (E, A) and corresponding consistency projector $\Pi_{(E,A)}$ and A^{diff}

$\Rightarrow$

$$x(t) = e^{A^{\text{diff}}(t-t_0)}\Pi_{(E,A)}x(t_0-) \in \mathfrak{C} \quad t \in (t_0, \infty).$$



Remark: At t_0 the presence of Dirac-impulses is possible!

Solution behavior for switched DAEs

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) \qquad \textbf{(swDAE)}$$

with consistency projectors Π_k and A_k^{diff}

Theorem (Impulse freeness, T. 2009)

All solutions of (**swDAE**) *are impulse free, if*

$$\forall k \in \{1,2,\ldots,M\}: \quad E_k(I-\Pi_k)\Pi_{k-1} = 0, \qquad \textbf{(IFC)}$$

where $\Pi_{-1} := \Pi_M$.

Corollary

All solutions of (**swDAE**) *satisfying* (**IFC**) *are given by*

$$x(t) = e^{A_k^{\text{diff}}(t-t_i)}\Pi_i e^{A_{i-1}^{\text{diff}}(t_i-t_{i-1})}\Pi_{i-1}\cdots e^{A_2^{\text{diff}}(t_3-t_2)}\Pi_2 e^{A_1^{\text{diff}}(t_2-t_1)}\Pi_1 x(t_1-)$$

Inhalt

Condition on consistency projectors

Assumption: commutative projectors

$$\forall i,j \in \{1,\dots,M\}: \quad \Pi_i\Pi_j = \Pi_j\Pi_i \qquad \textbf{(C)}$$

Lemma

(C) $\Rightarrow$ $\operatorname{im}\Pi_1\Pi_2\cdots\Pi_M = \operatorname{im}\Pi_1 \cap \operatorname{im}\Pi_2 \cap \ldots \cap \operatorname{im}\Pi_M$

Remark: $\operatorname{im}\Pi_1 \cap \ldots \cap \operatorname{im}\Pi_M = \mathfrak{C}_1 \cap \ldots \cap \mathfrak{C}_M$ and obviously the averaged system, if it exists, can only have solutions within the intersection of the consistency spaces, hence the projector

$$\Pi_\cap := \Pi_1\Pi_2\cdots\Pi_M$$

plays a crucial role!

In the example it was: $\Pi_1\Pi_2 = \Pi_1 \neq \Pi_2 = \Pi_2\Pi_1$

Main result

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) \qquad \textbf{(swDAE)}$$

$$\forall i,j \in \{1,\ldots,M\}: \quad \Pi_i\Pi_j = \Pi_j\Pi_i \qquad \textbf{(C)}$$

Theorem (Averaging for switched DAEs)

Consider impulse free (**swDAE**) *with consistency projectors* $\Pi_1,\ldots,\Pi_M$ *satisfying* (**C**) *and* $A_1^{\text{diff}},\ldots,A_M^{\text{diff}}$. *The averaged system is*

$$\dot{x}_{\text{av}} = \Pi_\cap A_{\text{av}}^{\text{diff}}\Pi_\cap x_{\text{av}}, \quad x_{\text{av}}(0) = \Pi_\cap x(0-)$$

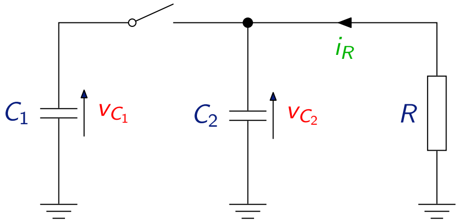
where $\Pi_\cap = \Pi_1\Pi_2\cdots\Pi_M$ *and*

$$A_{\text{av}}^{\text{diff}} := d_1A_1^{\text{diff}} + d_2A_2^{\text{diff}} + \ldots + d_MA_M^{\text{diff}}.$$

Then $\forall t \in (0,T]$

$$\|x(t) - x_{\text{av}}(t)\| = O(p)$$

Example



Switch independent: $0 = v_{C_2} - R i_R$

Switch dependent:

open	closed
$C_1 \dot v_{C_1} = 0,$	$C_1 \dot v_{C_1} + C_2 \dot v_{C_2} = -i_R,$
$C_2 \dot v_{C_2} = -i_R,$	$0 = v_{C_1} - v_{C_2},$

$\Rightarrow$ switched DAE $E_\sigma \dot x = A_\sigma x$ with $x = (v_{C_1}, v_{C_2}, i_R)^\top$ given by

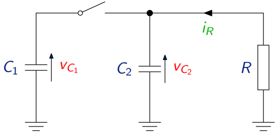
$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 & 0 \\ C_1 & 0 & 0 \\ 0 & C_2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \text{-}1 & \text{-}R \\ 0 & 0 & 0 \\ 0 & 0 & \text{-}1 \end{bmatrix} \right)$$

$$(E_2, A_2) = \left(\begin{bmatrix} 0 & 0 & 0 \\ C_1 & C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \text{-}R \\ 0 & 0 & 1 \\ 1 & \text{-}1 & 0 \end{bmatrix} \right)$$

Example

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 & 0 \\ C_1 & 0 & 0 \\ 0 & C_2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & -R \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right)$$

$$(E_2, A_2) = \left(\begin{bmatrix} 0 & 0 & 0 \\ C_1 & C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -R \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \right)$$



$\Rightarrow$ consistency projectors

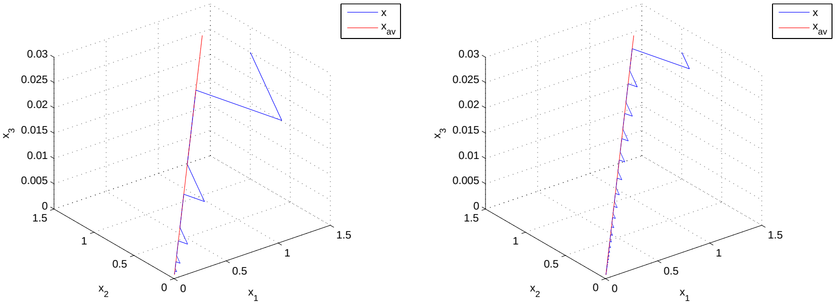
$$\Pi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{R} & 0 \end{bmatrix}, \quad \Pi_2 = \frac{1}{C_1+C_2} \begin{bmatrix} C_1 & C_2 & 0 \\ C_1 & C_2 & 0 \\ \frac{C_1}{R} & \frac{C_2}{R} & 0 \end{bmatrix}.$$

and **(C)** holds:

$$\Pi_1\Pi_2 = \Pi_2 = \Pi_2\Pi_1$$

Simulation results

$d_1 = 0.4, \quad p = 0.1$ $\qquad$ $d_1 = 0.4, \quad p = 0.02$



Summary

- Generalization of classical averaging result to switched DAEs
  - averaged system does not exist in all cases
  - Additional condition for consistency projectors necessary
  - classical averaged matrix must be projected to the right space
- Open questions
  - Commutativity of consistency projectors necessary?
  - Impulses: Convergence in the sense of distributions?