Observability of switched differential-algebraic equations for general switching signals

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51st IEEE Conference on Decision and Control
Tuesday, December 11, 2012, 10:40–11:00, Maui, USA

TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

Contents

Switched DAEs

DAE = Differential algebraic equation

Switched linear DAE (swDAE)

$$\begin{aligned} E_{\sigma(t)}\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) &= C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t) \end{aligned} \quad \text{or short} \quad \begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned}$$

with

- known switching signal $\sigma : \mathbb{R} \to \{1, 2, \ldots, \mathrm{p}\} =: \overline{\mathrm{p}}$
  - piecewise constant
  - locally finite jumps
- matrix tuples $(E_1, A_1, B_1, C_1, D_1), \ldots, (E_\mathrm{p}, A_\mathrm{p}, B_\mathrm{p}, C_\mathrm{p}, D_\mathrm{p})$
  - $E_p, A_p \in \mathbb{R}^{n\times n}$, $B_p \in \mathbb{R}^{n\times r}$, $C_p \in \mathbb{R}^{m\times n}$, $D_p \in \mathbb{R}^{m\times r}$, $p \in \overline{\mathrm{p}}$
  - (E_p, A_p) regular, i.e. $\det(E_p s - A_p) \not\equiv 0$, $p \in \overline{\mathrm{p}}$

Motivation

Why switched DAEs?

- First principles models often contain differential and algebraic equations $\to$ DAEs (instead of ODEs)
- Presence of switches (electrical circuits) or valves (water distribution networks) $\to$ switched DAEs
- Component faults $\to$ sudden changes in system description $\to$ switched DAEs

Observability

Determine internal states without putting sensors everywhere

- Power grid: monitor power flows through lines
- Water distribution: monitor pressures in tubes
- Fault detection

Fundamental system property: Observability

Global Observability of Switched DAEs

u —— **System** known σ unknown x —— y

Definition (Global observability)

The **(swDAE)** is (globally) observable $:\Leftrightarrow$
$\forall$ solutions $(u_1, x_1, y_1), (u_2, x_2, y_2)$: $\quad (u_1, y_1) \equiv (u_2, y_2) \;\Rightarrow\; x_1 \equiv x_2$

Proposition (0-distinguishability)

The **(swDAE)** *is observable if, and only if,*

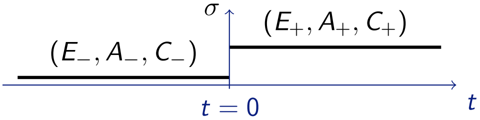
$$y \equiv 0 \text{ and } u \equiv 0 \quad \Rightarrow \quad x \equiv 0.$$

Hence consider in the following **(swDAE)** without inputs:

$$\boxed{\begin{aligned} E_\sigma \dot{x} &= A_\sigma x \\ y &= C_\sigma x \end{aligned}} \quad \text{and observability question:} \quad \boxed{y \equiv 0 \overset{?}{\Rightarrow} x \equiv 0}$$

Contents

The single switch result



Theorem (Unobservable subspace, Tanwani & T. 2010)

For **(swDAE)** *with* **a single switch** *the following equivalence holds*

$$x(0-) \in \mathcal{M} \quad \Leftrightarrow \quad y \equiv 0$$

where

$$\boxed{\mathcal{M} := \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\text{imp}-}}$$

Note that: $\quad x(0-) = 0 \;\Leftrightarrow\; x \equiv 0$

What are these four subspace?

The four subspaces

Unobservable subspace: $\mathcal{M} := \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\text{imp}-}$, i.e.

$$x(0-) \in \mathcal{M} \quad \Leftrightarrow \quad y_{(-\infty,0)} \equiv 0 \ \wedge \ y[0] = 0 \ \wedge \ y_{(0,\infty)} \equiv 0$$

The four spaces

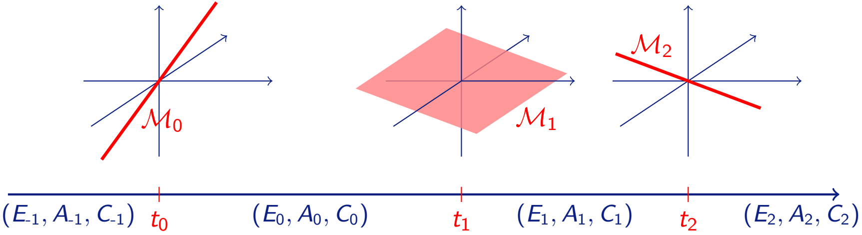
- Consistency: $x(0-) \in \mathfrak{C}_-$
- Left unobservability: $y_{(-\infty,0)} \equiv 0 \ \Leftrightarrow \ x(0-) \in \ker O_-$
- Right unobservability: $y_{(0,\infty)} \equiv 0 \ \Leftrightarrow \ x(0-) \in \ker O_+^-$
- Impulse unobervability: $y[0] = 0 \ \Leftrightarrow \ x(0-) \in \ker O_+^{\text{imp}-}$

These subspaces can be calculated (e.g. via the Wong sequences)

Multiple switchings

For convenience let $\sigma(t) = \begin{cases} -1, & \text{for } t < t_0 = 0, \\ k, & \text{for } t \in [t_k, t_{k+1}) \end{cases}$

Let $\mathcal{M}_k$ be the (local) unobservable subspace at k-th switch



Non-Necessity and Non-Sufficiency

$\mathcal{M}_k = \{0\}$ for some k not necessary for global observability!
$\mathcal{M}_k = \{0\}$ for some $k > 0$ not sufficient for global observability!

Flow between switches

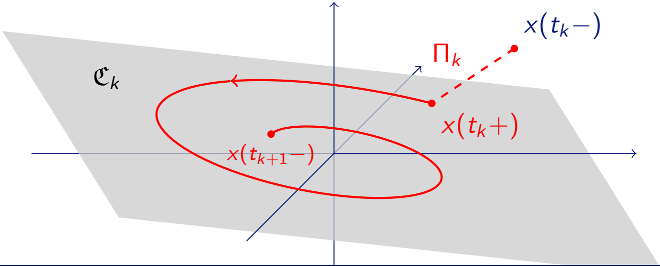
Characterization of observability

Need to consider the dynamics between the switches!

Theorem (Solution characterization)

Consider fixed mode k given by $E_k\dot{x} = A_k x$ with regular matrix pair $(E_k, A_k) \Rightarrow$ $\exists$ consistency projector Π_k $\exists$ flow matrix A_k^{diff}:

$$x(t) = e^{A_k^{\mathrm{diff}}(t-t_k)}\Pi_k x(t_k-) \in \mathfrak{C}_k \quad t \in [t_k, t_{k+1}).$$



Application to unobservable spaces

Assume $y \equiv 0 \Rightarrow x(t_k-) \in \mathcal{M}_k \; \forall k \in \mathbb{N}$

Backpropagation of knowledge

Use unobservable spaces from later switches to get information on earlier switches. One step:

$$\begin{aligned} x(t_{k+1}-) \in \mathcal{M}_{k+1} \;\Rightarrow\; & x(t_k+) \in e^{-A_k^{\text{diff}}\Delta_k}\mathcal{M}_{k+1} \\ \Rightarrow\; & x(t_k-) \in \Pi_k^{-1}(e^{-A_k^{\text{diff}}\Delta_k}\mathcal{M}_{k+1}) \end{aligned}$$

Hence improved knowledge for $x(t_k-)$:

$$x(t_k-) \in \mathcal{M}_k \cap \Pi_k^{-1}(e^{-A_k^{\text{diff}}\Delta_k}\mathcal{M}_{k+1})$$

$\Delta_k := t_{k+1} - t_k$

Main result

Consider switched DAE

(swDAE) $E_\sigma \dot{x} = A_\sigma x$

with fixed σ, switching times t_k, interval length Δ_k, corresponding consistency projectors Π_k and flow matrices A_k^{diff}.

Definition (Unobservable spaces of m-th order)

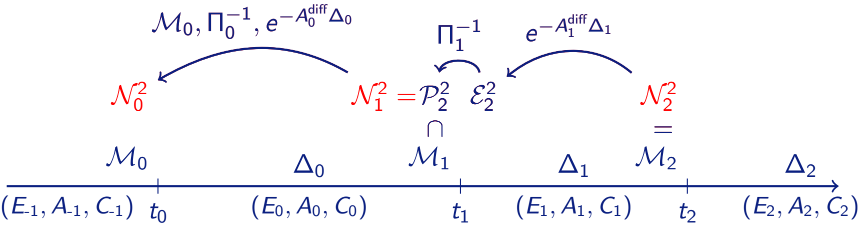
For $m \in \mathbb{N}$ let

$$\mathcal{N}_m^m := \mathcal{M}_m$$
$$\mathcal{N}_k^m := \mathcal{M}_k \cap \Pi_k^{-1}\left(e^{-A_k^{\text{diff}}\Delta_k}\mathcal{N}_{k+1}^m\right), k = m-1, \dots, 0$$

Theorem (Main result)

(swDAE) *is observable* $\quad\Leftrightarrow\quad \exists m \in \mathbb{N}: \quad \mathcal{N}_0^m = \{0\}$

Illustration of this result



Drawbacks

- Exact knowledge of switching signals necessary
- New switching $\rightarrow$ completely new calculation necessary

Improvements

Invariant subspaces

With the help of A^{diff}-invariant subspaces, obtain

- necessary condition for observability
- sufficient condition for observability

depending only on the mode sequence (and not on the switching times)

Determinability

(swDAE) determinable $:\Leftrightarrow$ $x_{[T,\infty)}$ can be determined for some $T > 0$

$\Leftrightarrow \quad \exists m \in \mathbb{N}$: $\mathcal{Q}^m = \{0\}$, where

$$\mathcal{Q}^0 := \Pi_0 \mathcal{M}_0$$
$$\mathcal{Q}^{k+1} := \Pi_{k+1}\left(\mathcal{M}_{k+1} \ \cap \ e^{A_k^{\text{diff}}\Delta_k}\mathcal{Q}^k\right), \quad k = 0, 1, 2, \ldots$$

$$\Pi_0\mathcal{M}_0 = \mathcal{Q}^0 \xrightarrow{e^{A_0^{\text{diff}}\Delta_0},\, \mathcal{M}_1,\, \Pi_1} \mathcal{Q}^1 \xrightarrow{e^{A_1^{\text{diff}}\Delta_1},\, \mathcal{M}_2,\, \Pi_2} \mathcal{Q}^2 \xrightarrow{e^{A_2^{\text{diff}}\Delta_2},\, \mathcal{M}_3,\, \Pi_3} \mathcal{Q}^3 \quad \cdots$$

Conclusions

- Considered switched DAEs

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned}$$

- local unobservable subspaces (single switch result)

$$\mathcal{M}_k = \mathfrak{C}_{k-1} \cap \ker O_{k-1} \cap \ker O_k^- \cap \ker O_k^{\text{imp}-}$$

- Characterization of global observability
- Sufficient and necessary conditions for observability only depending on mode sequence
- Determinability characterization $\rightarrow$ more suitable for observer design (future work)