The bang-bang funnel controller

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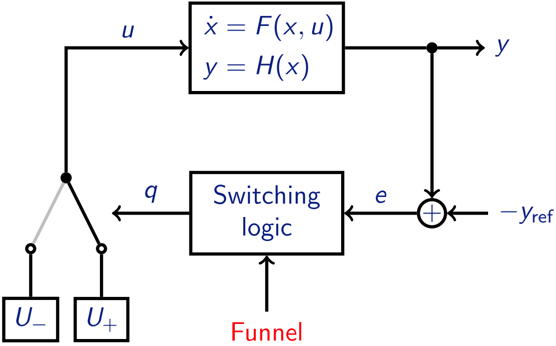
Content

1. Introduction
2. Relative degree one case
3. Higher relative degree

Feedback loop



Reference signal $y_{\text{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ suficiently smooth

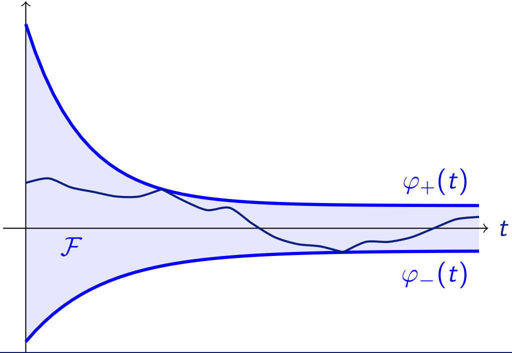
The funnel

Control objective

Error $e := y - y_{\text{ref}}$ evolves within *funnel*

$$\mathcal{F} = \mathcal{F}(\varphi_-, \varphi_+) := \{\ (t, e)\ \mid \varphi_-(t) \leq e \leq \varphi_+(t)\ \}$$

where $\varphi_\pm : \mathbb{R}_{\geq 0} \to \mathbb{R}$ sufficiently smooth



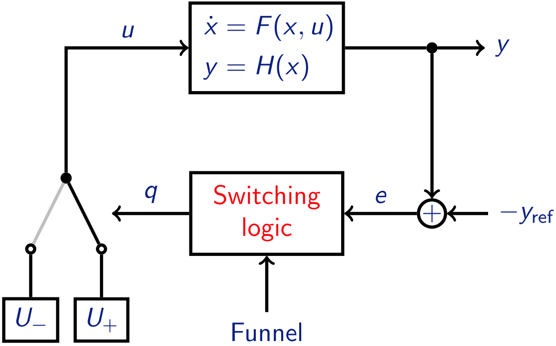
- time-varying strict error bound
- transient behaviour
- practical tracking ($|e(t)| < \lambda$ for $t >> 0$)

The bang-bang funnel controller

Continuous Funnel Controller: Introduced by Ilchmann et al. in 2002

New approach

Achieve control objectives with bang-bang control, i.e. $u(t) \in \{U_-, U_+\}$



Relative degree one

Definition (Relative degree one)

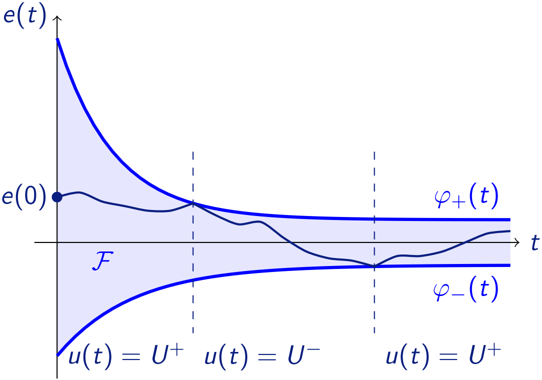
$$\begin{aligned} \dot{x} &= F(x,u) \\ y &= H(x) \end{aligned} \qquad \cong \qquad \begin{aligned} \dot{y} &= f(y,z) + \overbrace{g(y,z)}^{>0} u \\ \dot{z} &= h(y,z) \end{aligned}$$

- Structural assumption
- f, g, h can be unknown
- feasibility assumption (later) in terms of f, g, h and funnel

Important property

$$\begin{aligned} u(t) << 0 \quad &\Rightarrow \quad \dot{y}(t) << 0 \\ u(t) >> 0 \quad &\Rightarrow \quad \dot{y}(t) >> 0 \end{aligned}$$

Switching logic



Feasibility assumptions

$$\dot{y} = f(y,z) + g(y,z)u, \qquad y_0 \in \mathbb{R}$$
$$\dot{z} = h(y,z), \qquad z_0 \in Z_0 \subseteq \mathbb{R}^{n-1}$$

$$Z_t := \left\{ z(t) \;\middle|\; \begin{array}{l} z : [0,t] \to \mathbb{R}^{n-1} \text{ solves } \dot{z} = h(y,z) \text{ for some} \\ z^0 \in Z_0 \text{ and for some } y : [0,t] \to \mathbb{R} \\ \text{with } \varphi_-(\tau) \leq y(\tau) - y_{\text{ref}}(\tau) \leq \varphi_+(\tau) \\ \forall \tau \in [0,t] \end{array} \right\}.$$

Feasibility assumption

$$\forall t \geq 0 \ \forall z_t \in Z_t : \quad \begin{array}{l} U_- < \dfrac{\dot{\varphi}_+(t) + \dot{y}_{\text{ref}}(t) - f(y_{\text{ref}}(t) + \varphi_+(t), z_t)}{g(y_{\text{ref}}(t) + \varphi_+(t), z_t)} \\ U_+ > \dfrac{\dot{\varphi}_-(t) + \dot{y}_{\text{ref}}(t) - f(y_{\text{ref}}(t) + \varphi_-(t), z_t)}{g(y_{\text{ref}}(t) + \varphi_-(t), z_t)} \end{array}$$

Main result relative degree one

Theorem (Bang-bang funnel controller, Liberzon & T. 2010)

Relative degree one & Funnel & simple switching logic & Feasibility

$\Rightarrow$

Bang-bang funnel controller works:

- *existence and uniqueness of global solution*
- *error remains within funnel for all time*
- *no zeno behaviour*

Necessary knowledge:

- for controller implementation:
  - relative degree (one)
  - signals: error $e(t)$ and funnel boundaries $\varphi_\pm(t)$
- to check feasibility:
  - bounds on zero dynamics
  - bounds on f and g
  - bounds on y_{ref} and $\dot{y}_{\text{ref}}$
  - bounds on the funnel

Content

1 Introduction

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Relative degree r

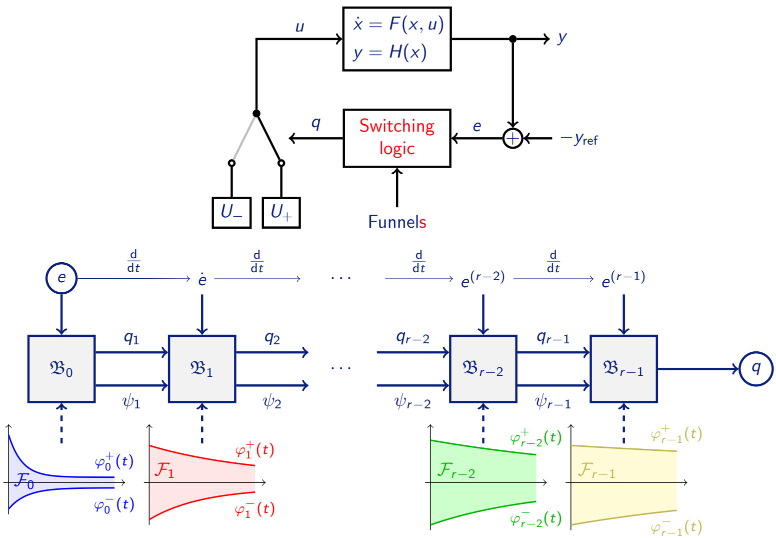
Definition (Relative degree r)

$$\begin{aligned} \dot{x} &= F(x,u) \\ y &= H(x) \end{aligned} \quad \cong \quad \begin{aligned} y^{(r)} &= f(y,\dot{y},\ldots,y^{(r-1)},z) + \overbrace{g(y,\ldots,y^{(r-1)},z)}^{>0}\, u \\ \dot{z} &= h(y,\dot{y},\ldots,y^{(r-1)},z) \end{aligned}$$

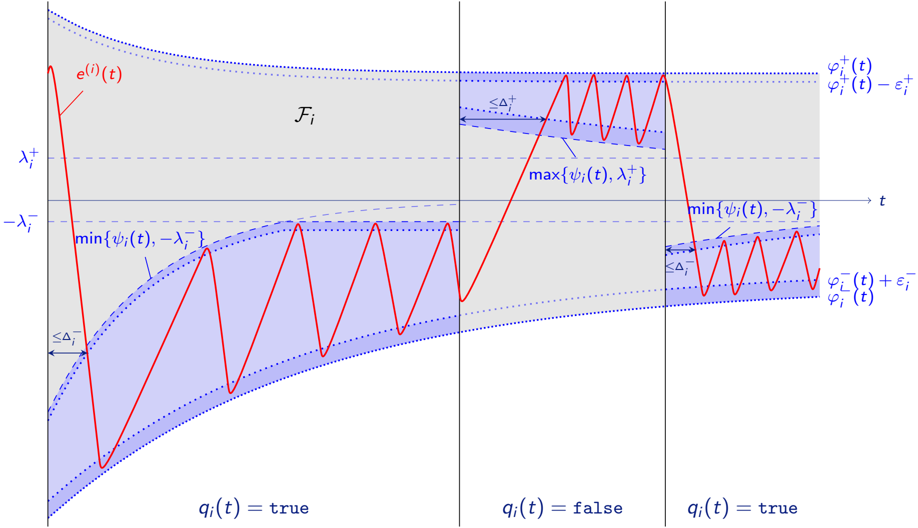
Essential property

$$u(t) << 0 \quad \Rightarrow \quad y^{(r)}(t) << 0$$
$$u(t) >> 0 \quad \Rightarrow \quad y^{(r)}(t) >> 0$$

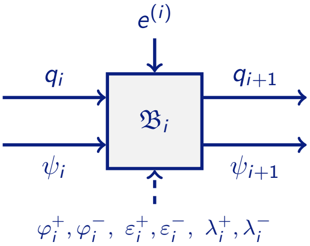
Hirachical structure of switching logic



Desired behaviour of block $\mathfrak{B}_i$



Definition of the swichting logic

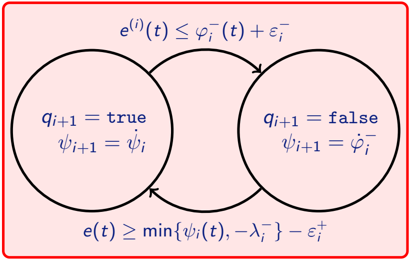


Goal of block $\mathfrak{B}_i$:

$q_i = \texttt{true} \quad \Rightarrow \quad \begin{cases} \text{make } e^{(i)} \text{ smaller} \\ \text{than } \min\{\psi_i, -\lambda_i^-\}, \end{cases}$

$q_i = \texttt{false} \quad \Rightarrow \quad \begin{cases} \text{make } e^{(i)} \text{ bigger} \\ \text{than } \max\{\psi_i, \lambda_i^+\} \end{cases}$

$q_1 = \texttt{true}$



$q_1 = \texttt{false}$

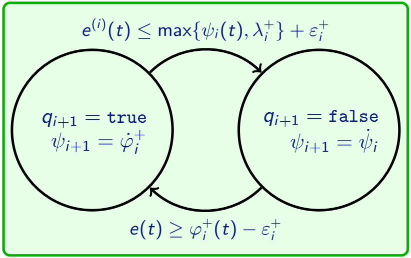
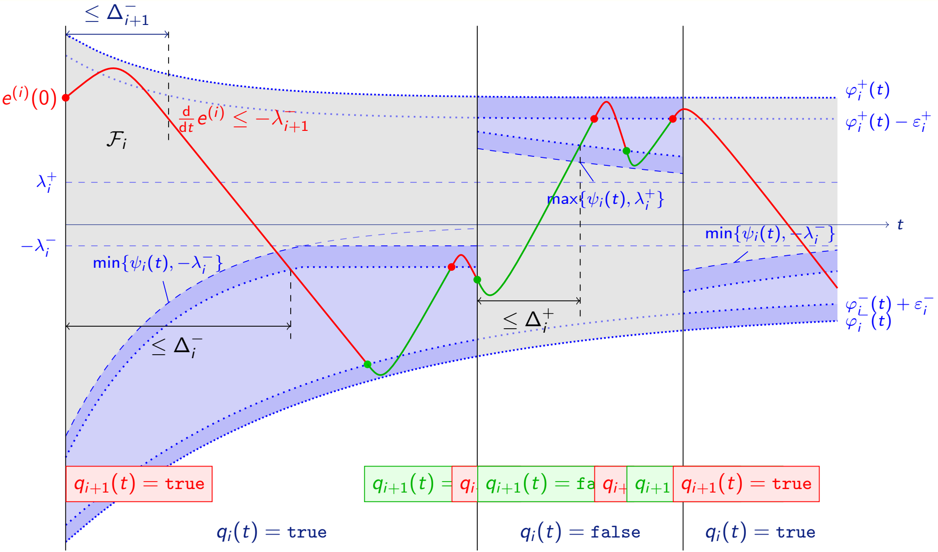


Illustration of switching logic



Main result

Theorem (Bang-bang funnel controller works, Liberzon & T. 2012)

Feasibility assumptions:

- *structural assumptions*
  - *relative degree r*
  - *smoothness and boundedness of y_{ref}*
- *funnels feasible*
  - *initial error values contained within funnels*
  - *sufficently smooth funnel boundaries*
  - *funnel boundaries large enough*
- *settling times and safety distance compatible*
- *U_+ and U_- large enough*

$\Rightarrow$ *bang-bang funnel controller works.*

Theorem (Feasibility)

Mild assumptions on $\mathcal{F}_0$ + BIBO of zero dynamics + boundedness of y_{ref}
$\Rightarrow$ *feasibility assumption satisfiable with sufficiently large U_+ and U_-*

Simulation for $r = 4$

Example (academic), possible finite escape time:

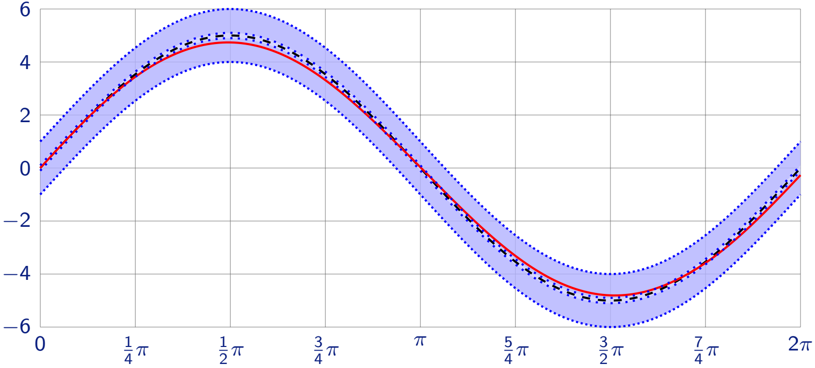
$$y^{(4)} = z\,\dddot{y}^2 + e^z u, \qquad y^{(i)}(0) = y_{\text{ref}}^{(i)}(0),\ \ i = 0,1,2,3,$$
$$\dot{z} = z(a-z)(z+b) - cy, \qquad z(0) = 0,$$
$$y_{\text{ref}}(t) = 5\sin(t)$$

control parameters (constant funnels):

$$\begin{aligned}
&\varphi_0^+ = -\varphi_0^- \equiv 1, && \varepsilon_0^+ = \varepsilon_0^- = 0.9, && && \Delta_0^+ = \Delta_0^- = \infty,\\
&\varphi_1^+ = -\varphi_1^- \equiv 0.5, && \varepsilon_1^+ = \varepsilon_1^- = 0.1, && \lambda_1^+ = \lambda_1^- = 0, && \Delta_1^+ = \Delta_1^- = \Delta_0^\pm/2 = \infty,\\
&\varphi_2^+ = -\varphi_2^- \equiv 0.5, && \varepsilon_2^+ = \varepsilon_2^- = 0.1, && \lambda_2^+ = \lambda_2^- = 0.2, && \Delta_2^+ = \Delta_2^- = 0.4,\\
&\varphi_3^+ = -\varphi_3^- \equiv 4.5, && \varepsilon_3^+ = \varepsilon_3^- = 0.1, && \lambda_3^+ = \lambda_3^- = 4, && \Delta_3^+ = \Delta_3^- = 0.1,\\
& && && \lambda_4^+ = \lambda_4^- = 102, && \Delta_4^+ = \Delta_4^- = 0.0001.
\end{aligned}$$

$$U_+ = -U_- = 254$$

Simulation results, tracking



Switching frequency: up to 1000 Hz
Number of switches in total: about 2200

Simulation results, error plots