# SWITCHED DAEs

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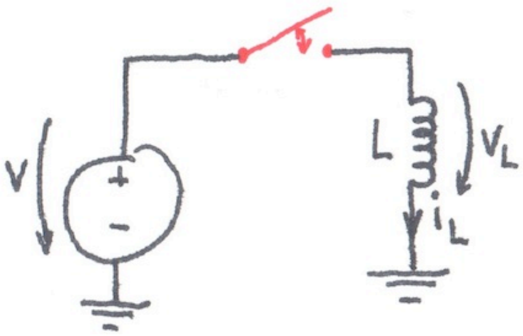
$$\boxed{\begin{aligned} E_{\sigma(t)}\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) &= C_{\sigma(t)}x(t) \end{aligned}} \qquad \text{(swDAE)}$$

Existence & Nature of solutions?

Jumps & Impulses!

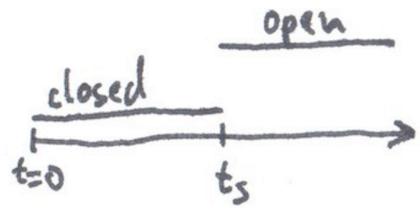
# Example



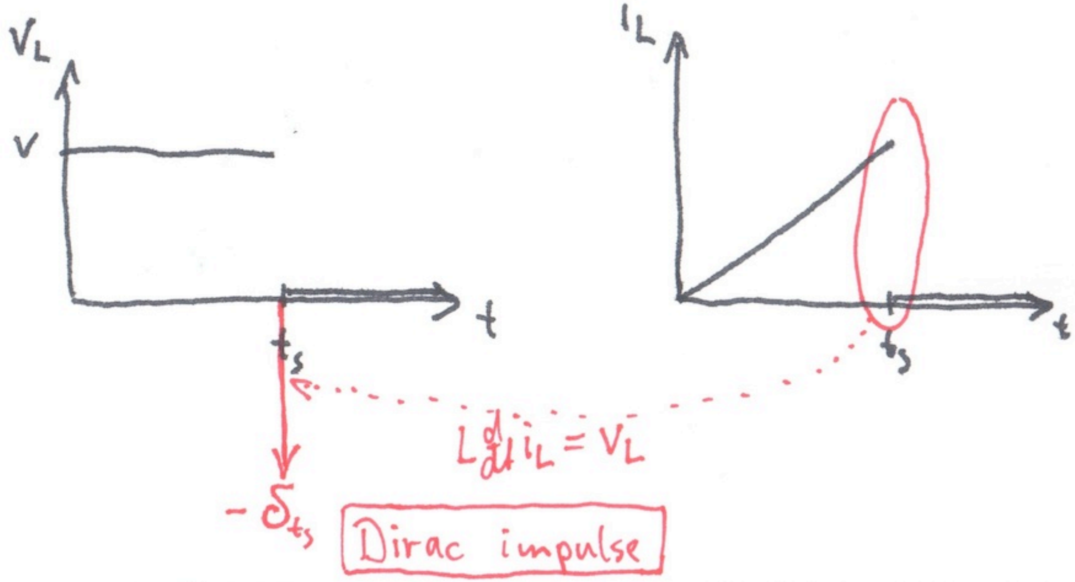

switch open: $L\frac{d}{dt}i_L = V_L$, $i_L = 0$ $\quad \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} i_L \\ V_L \end{pmatrix}^{\bullet} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} i_L \\ V_L \end{pmatrix}$

switch closed: $L\frac{d}{dt}i_L = V_L$, $V_L = V$ $\quad \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} i_L \\ V_L \end{pmatrix}^{\bullet} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{pmatrix} i_L \\ V_L \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Solution for $\sigma =$

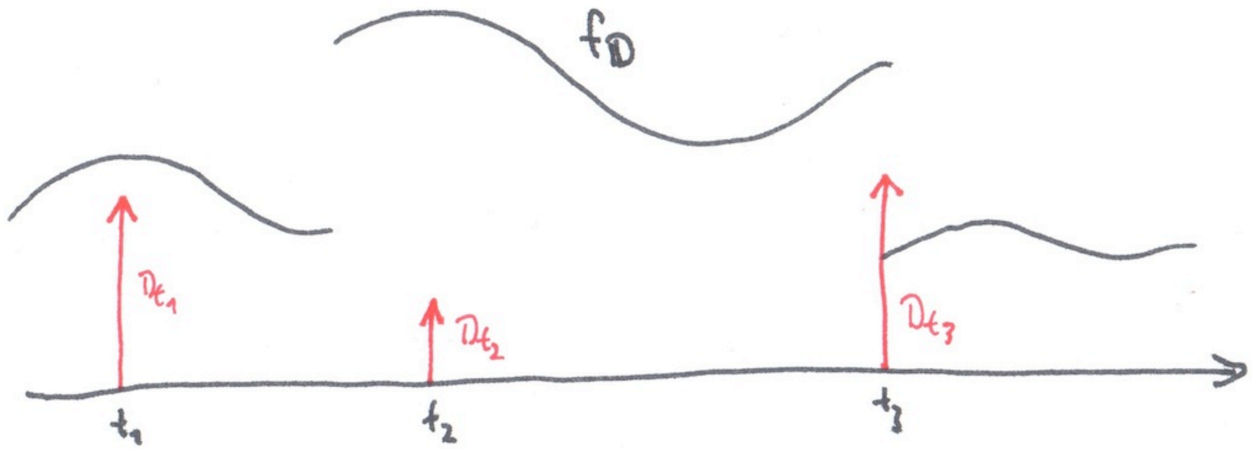



Assumptions: $v \equiv$ constant $\quad i(0) = 0$



Solutionspace: Piecewise-smooth distributions

$$\mathbb{D}_{pw\mathcal{C}^\infty} := \left\{ D = f_{\mathbb{D}} + \sum_{t\in T} D_t \;\middle|\; \begin{array}{l} f \in \mathcal{C}^\infty_{pw},\ T \subseteq \mathbb{R} \text{ lokally finite}, \\ \forall t \in T:\ D_t \in \operatorname{span}\{\delta_t, \delta_t', \delta_t'', \dots\} \end{array} \right\}$$




- Closed under differentiation
- Evaluation $D(t+) := f(t+)$, $D(t-) := f(t-)$
- Impulsive part $D[t] = D_t$ if $t \in T$, $D[t] = 0$ otherwise
- Multiplikation with $\mathcal{C}^\infty_{pw}$ well-defined $\longrightarrow$ Fuchssteiner multiplication

# Multiplication in $\mathbb{D}_{pw\mathcal{C}^\infty}$

Desired properties:

(M1) Algebra (i.e. $(F+G)H = FH + GH, \dots$)

(M2) Associativity: $(FG)H = F(GH)$

(M3) Differentiation rule: $(FG)' = F'G + FG'$

(M4) Functions: $(f \cdot g)_{\mathbb{D}} = f_{\mathbb{D}} \cdot g_{\mathbb{D}}$

(M5) Time invariance: $F(\cdot - t) \cdot G(\cdot - t) = (F \cdot G)(\cdot - t)$

## Theorem

$\exists$ <u>two</u> multiplications fulfilling (M1)–(M5) <u>characterized</u> by

either $\boxed{\mathbb{1}_{[0,\infty)} \delta_0 = \delta_0}$ or $\boxed{\mathbb{1}_{[0,\infty)} \delta_0 = 0}$

## Corollary

$$\delta^2 = 0$$

Proof: $0 = \left(\mathbb{1}_{[0,\infty)} \cdot \mathbb{1}_{(-\infty,0)}\right)' = \delta_0 \cdot \mathbb{1}_{(-\infty,0)} - \mathbb{1}_{[0,\infty)} \delta_0$

$0 = \mathrm{id} \cdot \delta_0 \cdot \mathbb{1}_{(-\infty,0)} = \mathrm{id} \cdot \mathbb{1}_{[0,\infty)} \delta_0 \Rightarrow \mathbb{1}_{[0,\infty)} \delta_0 = \alpha \delta_0$

$\Rightarrow \alpha \delta_0 = \mathbb{1}_{[0,\infty)} \delta_0 = \mathbb{1}_{[0,\infty)} \cdot \mathbb{1}_{[0,\infty)} \delta_0 = \mathbb{1}_{[0,\infty)} \alpha \delta_0 = \alpha^2 \delta_0 \Rightarrow \alpha = \alpha$

# Existence & Uniqueness of solutions

**Theorem**

(swDAE) uniquely solvable $\forall x(0-)$ and $\forall \sigma$

$\Leftrightarrow \forall p: (E_p, A_p)$ regular, i.e. $\det(sE_p - A_p) \not\equiv 0$

Quasi-Weierstrass-form

$$(E,A) \text{ regular} \Leftrightarrow \exists S,T \text{ inv. } (SET, SAT) = \left( \begin{bmatrix} I & \\ & N \end{bmatrix}, \begin{bmatrix} J & \\ & I \end{bmatrix} \right)$$

$N$ nilpotent

Consistency projector: $\Pi_{(E,A)} := T \begin{bmatrix} I & \\ & 0 \end{bmatrix} T^{-1}$

**Theorem** for $B_\sigma = 0$

$$x(t+) = \Pi_{(E_{\sigma(t+)}, A_{\sigma(t+)})} x(t-)$$