Modeling electrical circuits with switched differential algebraic equations

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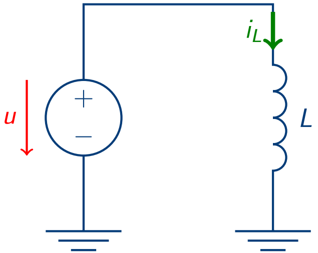
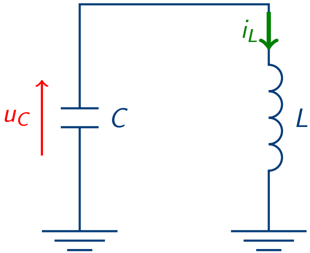
Standard modeling of circuits



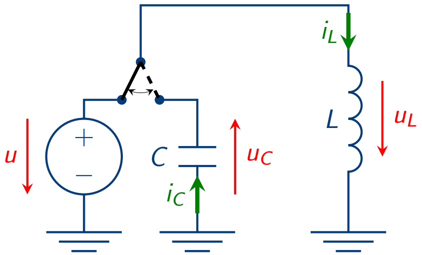
$$\frac{\mathrm{d}}{\mathrm{d}t} i_L = \frac{1}{L} u$$



$$\frac{\mathrm{d}}{\mathrm{d}t} i_L = -\frac{1}{L} u_C$$
$$\frac{\mathrm{d}}{\mathrm{d}t} u_C = \frac{1}{C} i_L$$

General form: $\dot{x} = Ax + Bu$

Switched ODE?



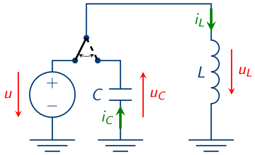
Mode 1: $\frac{d}{dt} i_L = \frac{1}{L} u$

Mode 2: $\frac{d}{dt} i_L = -\frac{1}{L} u_C$
$\frac{d}{dt} u_C = \frac{1}{C} i_L$

No switched ODE

Not possible to write as $\boxed{\dot{x}(t) = A_{\sigma(t)} x + B_{\sigma(t)} u}$.

Include algebraic equations in description



With $x := (i_L, u_L, i_C, u_C)$ write each mode as:

$$E_p \dot{x} = A_p x + B_p u$$

Algebraic equations $\Rightarrow$ E_p singular

Mode 1: $L\frac{\mathrm{d}}{\mathrm{d}t} i_L = u_L,\ C\frac{\mathrm{d}}{\mathrm{d}t} u_C = i_C,\ 0 = u_L - u,\ 0 = i_C$

$$\begin{bmatrix} L & 0 & 0 & 0 \\ 0 & 0 & 0 & C \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} u$$

Mode 2: $L\frac{\mathrm{d}}{\mathrm{d}t} i_L = u_L,\ C\frac{\mathrm{d}}{\mathrm{d}t} u_C = i_C,\ 0 = i_L - i_C,\ 0 = u_L + u_C$

Switched DAEs

DAE = Differential algebraic equation

Switched DAE

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \qquad \textbf{(swDAE)}$$

or short $E_\sigma \dot{x} = A_\sigma x + B_\sigma u$

with

- switching signal $\sigma : \mathbb{R} \to \{1, 2, \dots, \mathrm{p}\}$
  - piecewise constant
  - locally finite jumps
- modes $(E_1, A_1, B_1), \dots, (E_\mathrm{p}, A_\mathrm{p}, B_\mathrm{p})$
  - $E_p, A_p \in \mathbb{R}^{n\times n}$, $p = 1, \dots, \mathrm{p}$
  - $B_p : \mathbb{R}^{n\times m}$, $p = 1, \dots, \mathrm{p}$
- input $u : \mathbb{R} \to \mathbb{R}^m$

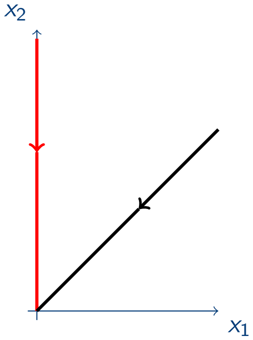
Question

Existence and nature of solutions?

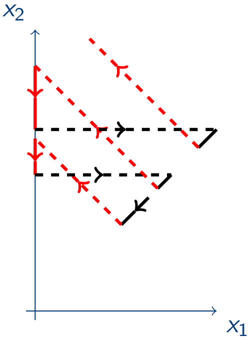
Simpler example

$$(E_1, A_1): \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} x \qquad (E_2, A_2): \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} x$$

non-switched:



switched:



Observations

Solutions

- Modes have constrained dynamics: Consistency spaces
- Switching $\Rightarrow$ Inconsistent initial values
- Inconsistent initial values $\Rightarrow$ Jumps in x

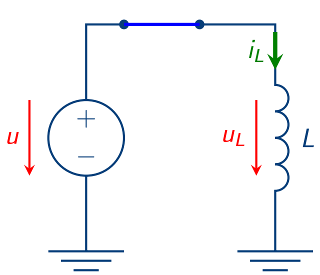
Stability

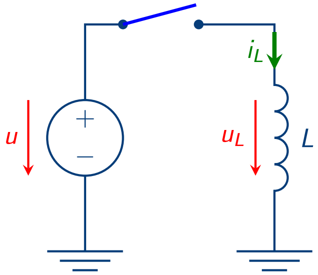
- Common Lyapunov function not sufficient
- Overall stability depend on jumps

Impulses

- Switching $\Rightarrow$ Dirac impulses in solution x
- Dirac impulse = infinite peak $\Rightarrow$ Instability

Impulse example

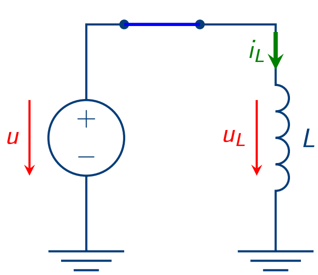


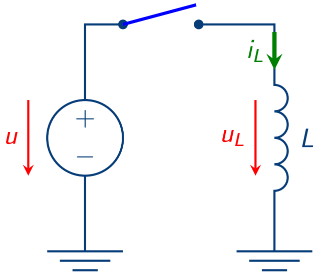


inductivity law: $L\frac{\mathrm{d}}{\mathrm{d}t}i_L = u_L$

switch dependent: $0 = u_L - u$ or $0 = i$

Impulse example





$x = [i_L, u_L]^\top$

$$\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u$$

$x = [i_L, u_L]^\top$

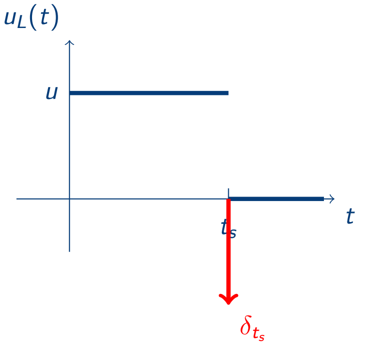
$$\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

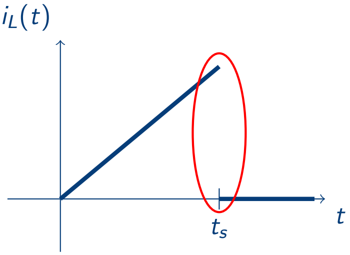
Solution of example

$L\frac{\mathrm{d}}{\mathrm{d}t} i_L = u_L, \qquad 0 = u_L - u$ or $0 = i_L$

Assume: u constant, $i_L(0) = 0$

switch at $t_s > 0$: $\sigma(t) = \begin{cases} 1, & t < t_s \\ 2, & t \geq t_s \end{cases}$





Contents

Distribution theorie - basic ideas

Distributions - overview

- Generalized functions
- Arbitrarily often differentiable
- Dirac-Impulse δ_0 is "derivative" of jump function $\mathbb{1}_{[0,\infty)}$

Two different formal approaches

1. Functional analytical: Dual space of the space of test functions (L. Schwartz 1950)
2. Axiomatic: Space of all "derivatives" of continuous functions (J. Sebastião e Silva 1954)

Distributions - formal

Definition (Test functions)

$\mathcal{C}_0^\infty := \{\ \varphi : \mathbb{R} \to \mathbb{R} \mid \varphi \text{ is smooth with compact support }\}$

Definition (Distributions)

$\mathbb{D} := \{\ D : \mathcal{C}_0^\infty \to \mathbb{R} \mid D \text{ is linear and continuous }\}$

Definition (Regular distributions)

$f \in L_{1,\mathrm{loc}}(\mathbb{R} \to \mathbb{R})$: $\quad f_\mathbb{D} : \mathcal{C}_0^\infty \to \mathbb{R},\ \varphi \mapsto \int_\mathbb{R} f(t)\varphi(t)\mathrm{d}t \in \mathbb{D}$

Definition (Derivative)

$D'(\varphi) := -D(\varphi')$

Dirac Impulse at $t_0 \in \mathbb{R}$

$\delta_{t_0} : \mathcal{C}_0^\infty \to \mathbb{R}, \quad \varphi \mapsto \varphi(t_0)$

Multiplication with functionen

Definition (Multiplication with smooth functions)

$\alpha \in \mathcal{C}^\infty : \quad (\alpha D)(\varphi) := D(\alpha\varphi)$

(swDAE) $\quad E_\sigma \dot{x} = A_\sigma x + B_\sigma u$

Coefficients not smooth

Problem: $E_\sigma, A_\sigma, B_\sigma \notin \mathcal{C}^\infty$

Observation:

$$\begin{matrix} E_\sigma \dot{x} = A_\sigma x + B_\sigma \\ i \in \mathbb{Z}: \;\; \sigma_{[t_i,t_{i+1})} \equiv p_i \end{matrix} \quad \Leftrightarrow \quad \forall i \in \mathbb{Z}: \; (E_{p_i}\dot{x})_{[t_i,t_{i+1})} = (A_{p_i}x + B_{p_i}u)_{[t_i,t_{i+1})}$$

New question: Restriction of distributions

Desired properties of distributional restriction

Distributional restriction:

$$\{\ M \subseteq \mathbb{R} \mid M \text{ interval }\ \} \times \mathbb{D} \to \mathbb{D}, \quad (M, D) \mapsto D_M$$

and for each interval $M \subseteq \mathbb{R}$

1. $D \mapsto D_M$ is a projection (linear and idempotent)
2. $\forall f \in L_{1,\mathrm{loc}} : \quad (f_{\mathbb{D}})_M = (f_M)_{\mathbb{D}}$
3. $\forall \varphi \in \mathcal{C}_0^\infty : \quad \begin{bmatrix} \operatorname{supp}\varphi \subseteq M & \Rightarrow & D_M(\varphi) = D(\varphi) \\ \operatorname{supp}\varphi \cap M = \emptyset & \Rightarrow & D_M(\varphi) = 0 \end{bmatrix}$
4. $(M_i)_{i\in\mathbb{N}}$ pairwise disjoint, $M = \bigcup_{i\in\mathbb{N}} M_i$:

$$D_{M_1 \cup M_2} = D_{M_1} + D_{M_2}, \quad D_M = \sum_{i\in\mathbb{N}} D_{M_i}, \quad (D_{M_1})_{M_2} = 0$$

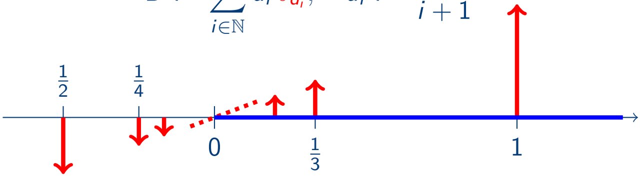
Theorem

Such a distributional restriction does not exist.

Proof of non-existence of restriction

Consider the following distribution(!):

$$D := \sum_{i \in \mathbb{N}} d_i \, \delta_{d_i}, \quad d_i := \frac{(-1)^i}{i+1}$$



Restriction should give

$$D_{(0,\infty)} = \sum_{k \in \mathbb{N}} d_{2k} \, \delta_{d_{2k}}$$

Choose $\varphi \in \mathcal{C}_0^\infty$ such that $\varphi_{[0,1]} \equiv 1$:

$$D_{(0,\infty)}(\varphi) = \sum_{k \in \mathbb{N}} d_{2k} = \sum_{k \in \mathbb{N}} \frac{1}{2k+1} = \infty$$

Dilemma

Switched DAEs

- Examples: distributional solutions
- Multiplication with non-smooth coefficients
- Or: Restriction on intervals

Distributions

- Distributional restriction not possible
- Multiplication with non-smooth coefficients not possible
- *Initial value problems cannot be formulated*

Underlying problem

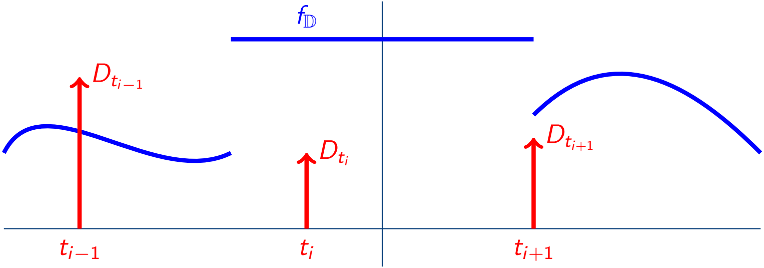
Space of distributions too big.

Piecewise smooth distributions

Define a suitable smaller space:

Definition (Piecewise smooth distributions $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$)

$$\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty} := \left\{ f_{\mathbb{D}} + \sum_{t\in T} D_t \;\middle|\; \begin{array}{l} f \in \mathcal{C}^\infty_{\mathrm{pw}}, \\ T \subseteq \mathbb{R} \text{ locally finite}, \\ \forall t \in T : D_t = \sum_{i=0}^{n_t} a_i^t \delta_t^{(i)} \end{array} \right\}$$



Properties of $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$

- $\mathcal{C}^\infty_{\text{pw}}$ "$\subseteq$" $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$
- $D \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty} \Rightarrow D' \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$
- Restriction $\mathbb{D}_{\text{pw}\mathcal{C}^\infty} \to \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$, $D \mapsto D_M$ for all intervals $M \subseteq \mathbb{R}$ well defined
- Multiplication with $\mathcal{C}^\infty_{\text{pw}}$-functions well defined
- Left and right sided evaluation at $t \in \mathbb{R}$: $D(t-), D(t+)$
- Impulse at $t \in \mathbb{R}$: $D[t]$

(swDAE) $E_\sigma \dot{x} = A_\sigma x + B_\sigma u$ with input $u \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^m$

Application to (swDAE)

x solves (swDAE) $:\Leftrightarrow$ $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ and (swDAE) holds in $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$

Relevant questions

Consider $E_\sigma \dot{x} = A_\sigma x + B_\sigma u$ with regular matrix pairs (E_p, A_p).

- Existence of solutions?
- Uniqueness of solutions?
- Inconsistent initial value problems?
- Jumps and impulses in solutions?
- Conditions for impulse free solutions?
- Stability

Theorem (Existence and uniqueness)

$\forall x^0 \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n \ \forall t_0 \in \mathbb{R} \ \forall u \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^m \ \exists! x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$:

$$x_{(-\infty,t_0)} = {x^0}_{(-\infty,t_0)}$$
$$(E_\sigma \dot{x})_{[t_0,\infty)} = (A_\sigma x + B_\sigma u)_{[t_0,\infty)}$$

Remark: x is called *consistent solution* $\quad :\Leftrightarrow \quad E_\sigma \dot{x} = A_\sigma x + B_\sigma u$

Contents

Regularity: Definition and characterization

Definition (Regularity)

(E, A) regular $\quad :\Leftrightarrow \quad \det(sE - A) \not\equiv 0$

Theorem (Characterizations of regularity)

The following statements are equivalent:

- *(E, A) is regular.*
- *$\exists S, T \in \mathbb{R}^{n\times n}$ invertible which yield quasi-Weierstrass form*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \qquad \textbf{(QWF)}$$

  where N is a nilpotent matrix.
- *$\forall$ smooth f $\exists$ classical solution x of $E\dot{x} = Ax + f$ which is uniquely given by $x(t_0)$ for any $t_0 \in \mathbb{R}$.*
- *x solves $E\dot{x} = Ax$ and $x(0) = 0 \quad \Rightarrow \quad x \equiv 0$.*

Wong sequences and the quasi-Weierstrass form

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \qquad \textbf{(QWF)}$$

Theorem ([Armentano '86], [Berger, Ilchmann, T. '10])

For regular (E, A) define the Wong sequences

$$\mathcal{V}^{i+1} := A^{-1}(E\mathcal{V}^i), \qquad \mathcal{V}^0 := \mathbb{R}^n,$$
$$\mathcal{W}^{i+1} := E^{-1}(A\mathcal{W}^i), \qquad \mathcal{W}^0 := \{0\}.$$

Then $\mathcal{V}^i \overset{finite}{\longrightarrow} \mathcal{V}^*$ and $\mathcal{W}^i \overset{finite}{\longrightarrow} \mathcal{W}^*$. Choose V, W such that $\operatorname{im} V = \mathcal{V}^*$ and $\operatorname{im} W = \mathcal{W}^*$ than

$$T := [V, W], \quad S := [EV, AW]^{-1}$$

*yield (**QWF**).*

Consistency projector

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right) \qquad \textbf{(QWF)}$$

Definition (Consistency projector)

Let (E, A) be regular with **(QWF)**, consistency projector:

$$\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$$

Theorem

x *solves* $E_\sigma \dot{x} = A_\sigma x \quad \Rightarrow \quad \forall t \in \mathbb{R}:$

$$x(t+) = \Pi_{(E_q,A_q)} x(t-), \quad q := \sigma(t+)$$

Differential projector

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \qquad \textbf{(QWF)}$$

Definition (Differential projector)

Let (E, A) be regular with **(QWF)**, differential projector:

$$\Pi_{(E,A)}^{\text{diff}} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S$$

$A^{\text{diff}} := \Pi_{(E,A)}^{\text{diff}} A$

Theorem

x *solves* $E_\sigma \dot{x} = A_\sigma x \quad \Rightarrow \quad \forall t \in \mathbb{R}:$

$$\dot{x}(t+) = A_{\sigma(t+)}^{\text{diff}} x(t+)$$

Impulse projector

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \qquad \textbf{(QWF)}$$

Definition (Impulse projector)

Let (E, A) be regular with **(QWF)**, impulse projector:

$$\Pi^{\text{diff}}_{(E,A)} := T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S$$

$E^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)} E$

Theorem

x *solves* $E_\sigma \dot{x} = A_\sigma x \quad \Rightarrow \quad \forall t \in \mathbb{R}:$

$$x[t] = \sum_{i=0}^{n-2} (E^{\text{imp}}_{\sigma(t+)})^{i+1} (x(t+) - x(t-)) \delta_t^{(i)}$$

Solution formula, inhomogeneous non-switched case

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \qquad \textbf{(QWF)}$$

$$\Pi_{(E,A)} := T \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] T^{-1}, \qquad \Pi^{\text{diff}}_{(E,A)} := T \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] S, \qquad \Pi^{\text{imp}}_{(E,A)} := T \left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right] S,$$

$$A^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)} A, \qquad E^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)} E$$

Theorem (Explicit solution formula, non-switched)

x *solves* $E\dot{x} = Ax + f \quad \Leftrightarrow \quad \exists c \in \mathbb{R}^n \; \forall t \in \mathbb{R}:$

$$x(t) = e^{A^{\text{diff}} t} \Pi_{(E,A)} c + \int_0^t e^{A^{\text{diff}}(t-s)} \Pi^{\text{diff}}_{(E,A)} f(s) \mathrm{d}s - \sum_{i=0}^{n-1} (E^{\text{imp}})^i \Pi^{\text{imp}}_{(E,A)} f^{(i)}(t)$$

Jumps and impulses for switched DAE

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u \qquad \textbf{(swDAE)}$$

$$B_q^{\text{imp}} := \Pi^{\text{imp}}_{(E_q,A_q)} B_q, \quad q \in \{1,\ldots,\mathrm{p}\}$$

Theorem (Jumps and impulses)

*x solves (**swDAE**)* $\Rightarrow \quad \forall t \in \mathbb{R}$:

$$x(t+) = \Pi_{(E_q,A_q)} x(t-) - \sum_{i=0}^{n-1} (E_q^{\text{imp}})^i B_q^{\text{imp}} u^{(i)}(t+),$$

$$x[t] = -\sum_{i=0}^{n-1} (E_q^{\text{imp}})^{i+1} (I - \Pi_{(E_q,A_q)}) x(t-)\ \delta_t^{(i)} \qquad q := \sigma(t+)$$

$$- \sum_{i=0}^{n-1} (E_q^{\text{imp}})^{i+1} \sum_{j=0}^{i} B_q^{\text{imp}} u^{(i-j)}(0+)\ \delta_t^{(j)}$$

Asymptotic stability

$$E_\sigma \dot{x} = A_\sigma x \qquad \textbf{(swDAEhom)}$$

Definition (Asymptotic stability)

(swDAEhom) asymptotically stable $:\Leftrightarrow$ $\forall$ solutions $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$:

(S) $\forall \varepsilon > 0\ \exists \delta > 0 : \quad \|x(0-)\| < \delta \ \Rightarrow \forall t > 0 : \ \|x(t\pm)\| < \varepsilon$,

(A) $x(t\pm) \to 0$ as $t \to \infty$,

(I) $\forall t \geq 0 : \quad x[t] = 0$.

Theorem (Impulse-freeness)

$\forall p, q \in \{1, \ldots, \mathrm{p}\} : \ E_q(I - \Pi_{(E_q,A_q)})\Pi_{(E_p,A_p)} = 0 \quad \Rightarrow \quad$ **(I)**

Lyapunov functions

Consider non-switched DAE

$$E\dot{x} = Ax$$

with consistency space $\mathcal{V}^*$

Definition (Lyapunov function for $E\dot{x} = Ax$)

$Q = Q^\top > 0$ on $\mathcal{V}^*$ and $P = P^\top > 0$ solves

$$A^\top PE + E^\top PA = -Q \quad \text{(generalized Lyapunov equation)}$$

Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0} : x \mapsto (Ex)^\top PEx$

$$\tfrac{\mathrm{d}}{\mathrm{d}t} V(x) = (E\dot{x})^\top PEx + (Ex)^\top PE\dot{x} = x^\top (A^\top PE + E^\top PA)x = -x^\top Qx$$

Theorem (Owens & Debeljkovic 1985)

$E\dot{x} = Ax$ asymptotically stable $\Leftrightarrow$ $\exists$ Lyapunov function

Stability under arbitrary switching

Consider $E_\sigma \dot{x} = A_\sigma x$ with additional assumption:

($\exists$V$_\mathbf{p}$): $\forall p \in \{1,\ldots,N\}$ $\exists$ Lyapunov function V_p for (E_p, A_p)

i.e. each DAE (E_p, A_p) is asymp. stable

(IFC): $\forall p,q \in \{1,\ldots,N\} \quad E_q(I - \Pi_{(E_q,A_q)})\Pi_{(E_p,A_p)} = 0$

Lyapunov jump condition

(LJC): $\forall p,q = 1,\ldots,N\ \forall x \in \mathfrak{C}_{(E_q,A_q)}: \quad V_p(\Pi_p x) \leq V_q(x)$

Theorem (Liberzon and T. 2009)

(IFC) $\wedge$ **($\exists$V$_\mathbf{p}$)** $\wedge$ **(LJC)** $\Rightarrow$ **(swDAE)** *asymptotically stable*

Slow switching

Slow switching signals with average dwell time $\tau_a > 0$:

$$\Sigma_{\tau_a} := \left\{ \sigma \in \Sigma \;\middle|\; \exists N_0 > 0 \; \forall t \in \mathbb{R} \; \forall \Delta t > 0 : \; N_\sigma(t, t + \Delta t) < N_0 + \frac{\Delta t}{\tau_a} \right\}.$$

where $N_\sigma(t_1, t_2)$ is the number of switches in interval $[t_1, t_2)$

Theorem (Liberzon & T. 2010)

$\exists \tau_a > 0 \; \forall \sigma \in \Sigma_{\tau_a}$: **(IFC)** $\wedge$ **($\exists$V$_p$)** $\Rightarrow$ **(swDAE)** *asymptotically stable*

Explicit formula for τ_a

It is possible to explicitly calculate τ_a in terms of minimum and maximum eigenvalues of certain matrices involving P_p, Q_p.

Conclusions

- DAEs natural for modeling electrical circuits
- Switches induce jumps and impulses $\Rightarrow$ Distributional solutions
  - General distributions not suitable
  - Smaller space: Piecewise-smooth distributions
- Regularity $\Leftrightarrow$ Existence & uniqueness of solutions
- Unique consistency jumps
- Condition for impulse-freeness
- Stability

Matlab Code for calculating the consistency projectors

Calculating a basis of the pre-image $A^{-1}(\operatorname{im} S)$:

```
function V=getPreImage(A,S)
[m1,n1]=size(A); [m2,n2]=size(S);
if m1==m2 | m2==0
    H=null([A,S]);
    V=colspace(H(1:n1,:));
end;
```

Calculating V with $\operatorname{im} V = \mathcal{V}_{k^*}$:

```
function V = getVspace(E,A)
[m,n]=size(E);
if (m==n) & size(E)==size(A)
    V=eye(n,n);
    oldsize=n; newsize=n; finished=0;
    while finished==0;
       EV=colspace(E*V);
       V=getPreImage(A,EV);
       oldsize=newsize;
       newsize=rank(V);
       finished = (newsize==oldsize);
    end;
end;
```

Calculating W with $\operatorname{im} W = \mathcal{W}_{k^*}$ analog.

Explicit formula for sufficient average dwell time

Let P_p, Q_p be the solutions of the generalized Lyapunov equation corresponding to (E_p, A_p), let O_p be an orthogonal basis matrix of $\mathcal{V}_p^*$ and let

$$\mu_{p,q} := \frac{\lambda_{\max}(O_p^\top \Pi_q^\top E_q^\top P_q E_q \Pi_q O_p)}{\lambda_{\min}(O_p^\top E_p^\top P_p E_p O_p)} > 0, \quad \lambda_p := \frac{\lambda_{\min}(O_p^\top Q_p O_p)}{\lambda_{\max}(O_p^\top E_p^\top P_p E_p O_p)} > 0,$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimal and maximal eigenvalue of a symmetric matrix, respectively. Then an average dwell time of

$$\tau_a > \frac{\max_{p,q} \ln \mu_{p,q}}{\min_p \lambda_p}$$

guarantees asymptotic stability.