# The bang-bang funnel controller

Stephan Trenn (joint work with Daniel Liberzon)

Institute for Mathematics, University of Würzburg, Germany

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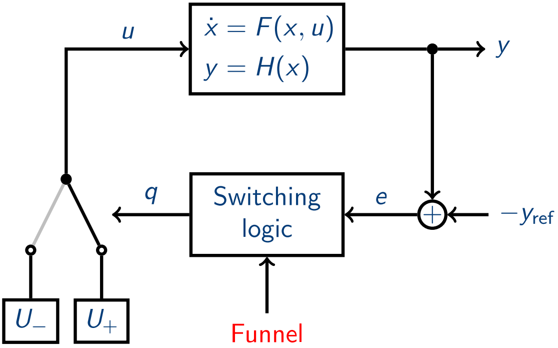
# Content

# Feedback loop




Reference signal $y_{\text{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ absolutely continuous

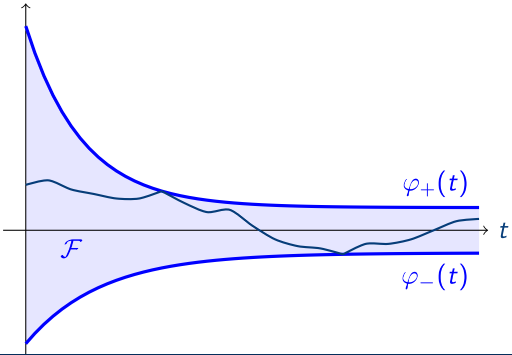
# The funnel

## Control objective

Error $e := y - y_{\text{ref}}$ evolves within *funnel*

$$\mathcal{F} = \mathcal{F}(\varphi_-, \varphi_+) := \{ \ (t, e) \ | \ \varphi_-(t) \leq e \leq \varphi_+ \ \}$$

where $\varphi_\pm : \mathbb{R}_{\geq 0} \to \mathbb{R}$ absolutely continuous




- time-varying strict error bound
- transient behaviour
- practical tracking ($|e(t)| < \lambda$ for $t >> 0$)

# Background on the continuous funnel controller

## Continuous funnel controller

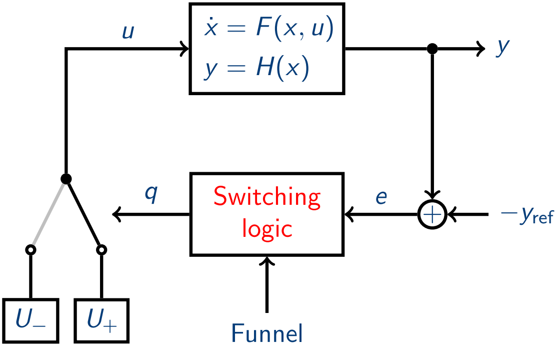
$$u(t) = -k(t)e(t)$$
$$k(t) = \frac{1}{\operatorname{dist}(e(t), \partial\mathcal{F}(t))} \overset{\varphi^+=\varphi=-\varphi^-}{=} \frac{1}{\varphi(t) - |e(t)|}$$

- Introduced by Ilchmann et al. in 2002, based on ideas from
  - high gain adaptive control ($\dot{k}(t) = e^2(t)$)
  - lambda tracking ($\dot{k}(t) = e^2(t)$ or $\dot{k}(t) = 0$ if $|e(t)| \leq \lambda$)
  - the work by Miller & Davison from 1991 ($k(t) = k_{\sigma(t)}$ with e.g. $k_i = (-3)^i$)
- Independent of the system's parameters & reference signal
- Guaranteed transient performance & practical tracking
- Disadvantages of the original funnel controller:
  - Only works for relative degree one systems
  - Input constraints $\Rightarrow$ feasibility assumptions

# The bang-bang funnel controller

## New approach

Achieve control objectives with bang-bang control, i.e. $u(t) \in \{U_-, U_+\}$



# Relative degree one

## Definition (Relative degree one)

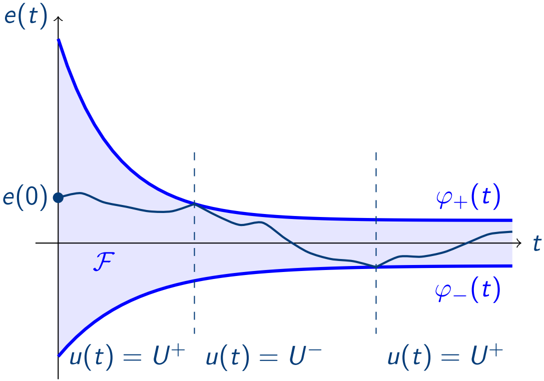
$$\begin{aligned} \dot{x} &= F(x,u) \\ y &= H(x) \end{aligned} \qquad \cong \qquad \begin{aligned} \dot{y} &= f(y,z) + \overbrace{g(y,z)}^{>0} u \\ \dot{z} &= h(y,z) \end{aligned}$$

- Structural assumption
- $f, g, h$ can be unknown
- feasibility assumption (later) in terms of $f, g, h$ and funnel

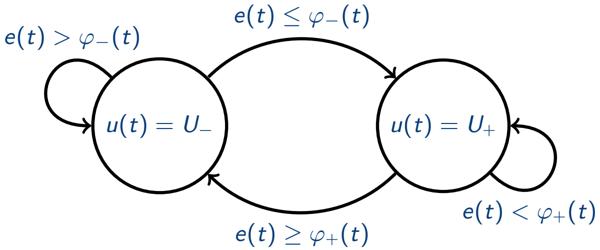
## Important property

$$\begin{aligned} u(t) << 0 \quad &\Rightarrow \quad \dot{y}(t) << 0 \\ u(t) >> 0 \quad &\Rightarrow \quad \dot{y}(t) >> 0 \end{aligned}$$

# Switching logic



# Switching logic




**Too simple?**

$\Rightarrow$ Feasibility assumptions

# Feasibility assumptions

$$\begin{aligned} \dot{y} &= f(y,z) + g(y,z)u, && y_0 \in \mathbb{R} \\ \dot{z} &= h(y,z), && z_0 \in Z_0 \subseteq \mathbb{R}^{n-1} \end{aligned}$$

$$Z_t := \left\{ z(t) \;\middle|\; \begin{array}{l} z : [0,t] \to \mathbb{R}^{n-1} \text{ solves } \dot{z} = h(y,z) \text{ for some} \\ z^0 \in Z_0 \text{ and for some } y : [0,t] \to \mathbb{R} \\ \text{with } \varphi_-(\tau) \leq y(\tau) - y_{\text{ref}}(\tau) \leq \varphi_+(\tau) \\ \forall \tau \in [0,t] \end{array} \right\}.$$

## Feasibility assumption

$$\forall t \geq 0 \; \forall z_t \in Z_t : \quad \begin{array}{l} U_- < \dfrac{\dot{\varphi}_+(t) + \dot{y}_{\text{ref}}(t) - f(y_{\text{ref}}(t) + \varphi_+(t), z_t)}{g(y_{\text{ref}}(t) + \varphi_+(t), z_t)} \\ U_+ > \dfrac{\dot{\varphi}_-(t) + \dot{y}_{\text{ref}}(t) - f(y_{\text{ref}}(t) + \varphi_-(t), z_t)}{g(y_{\text{ref}}(t) + \varphi_-(t), z_t)} \end{array}$$

# Main result relative degree one

**Theorem (Bang-bang funnel controller)**

*Relative degree one + Funnel & simple switching logic + Feasibility*
$\Rightarrow$
*Bang-bang funnel controller works:*

- *existence and uniqueness of global solution*
- *error remains within funnel for all time*
- *no zeno behaviour*

Necessary knowledge:

- for controller implementation:
  - relative degree (one)
  - signals: error $e(t)$ and funnel boundaries $\varphi_\pm(t)$
- to check feasibility:
  - bounds on zero dynamics
  - bounds on $f$ and $g$
  - bounds on $y_{\text{ref}}$ and $\dot{y}_{\text{ref}}$
  - bounds on the funnel

# Content

# Relative degree two

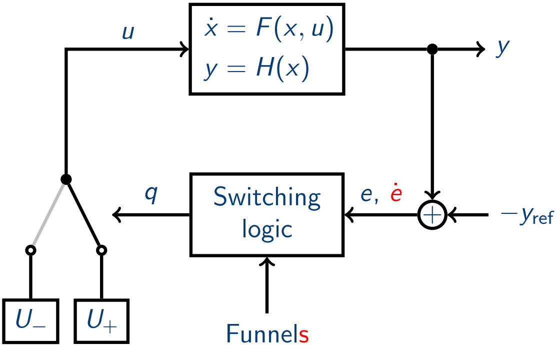
## Definition (Relative degree two)

$$\begin{aligned} \dot{x} &= F(x,u) \\ y &= H(x) \end{aligned} \qquad \cong \qquad \begin{aligned} \ddot{y} &= f(y,\dot{y},z) + \overbrace{g(y,\dot{y},z)}^{>0} u \\ \dot{z} &= h(y,\dot{y},z) \end{aligned}$$

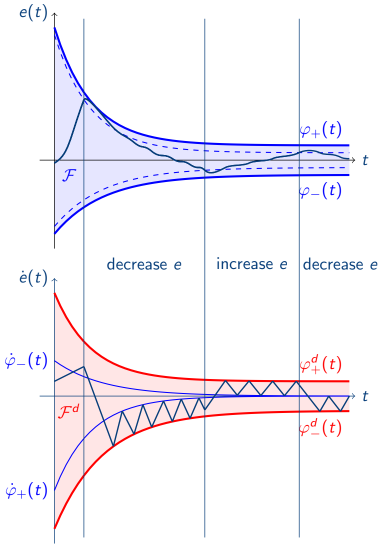
## Important property

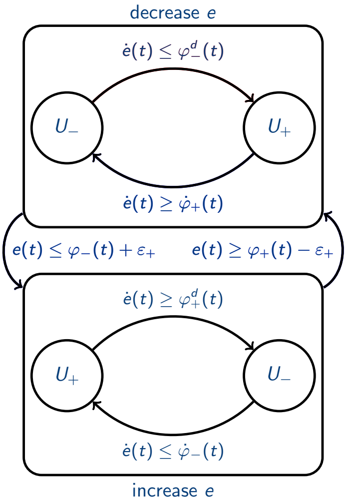
$$\begin{aligned} u(t) << 0 \quad &\Rightarrow \quad \ddot{y}(t) << 0 \\ u(t) >> 0 \quad &\Rightarrow \quad \ddot{y}(t) >> 0 \end{aligned}$$

# Feedback loop



# The switching logic



# Feasibility assumptions

Funnels $\mathcal{F}(\varphi_+, \varphi_-)$, $\mathcal{F}^d(\varphi_+^d, \varphi_-^d)$
Security distances $\varepsilon^+, \varepsilon^- > 0$

## Feasibility of funnels

- $\forall t \geq 0: \quad \varphi_+(t) - \varepsilon_+ > 0 \quad \text{and} \quad \varphi_-(t) + \varepsilon_- < 0$
- $\forall t \geq 0: \quad \varphi_+^d(t) > \dot{\varphi}_-(t) \quad \text{and} \quad \varphi_-^d(t) < \dot{\varphi}_+(t)$

$$\begin{aligned} \ddot{y} &= f(y, \dot{y}, z) + g(y, \dot{y}, z)u \\ \dot{z} &= h(y, \dot{y}, z) \end{aligned}$$

$Z_t := \{\ z(t)\ |\ z \text{ solves } \dot{z} = h(y, \dot{y}, z), z(0) \in Z_0\ \}$
Choose $\delta_\pm > 0$ such that

$$\begin{aligned} \delta_+ &> \max\{\dot{\varphi}_-^d(t), \ddot{\varphi}_-(t)\} \quad \text{and} \\ -\delta_- &< \min\{\dot{\varphi}_+^d(t), \ddot{\varphi}_+(t)\} \quad \forall t \geq 0 \end{aligned}$$

# Feasibility assumptions

## Feasibility assumption 1

$$U_- < \frac{-\delta_- + \ddot{y}_{\text{ref}}(t) + f(y_t, \dot{y}_t, z_t)}{g(y_t, \dot{y}_t, z_t)},$$
$$U_+ > \frac{\delta_+ + \ddot{y}_{\text{ref}}(t) + f(y_t, \dot{y}_t, z_t)}{g(y_t, \dot{y}_t, z_t)},$$

$\forall t \geq 0, \quad \forall y_t \in [y_{\text{ref}}(t) + \varphi_-(t), y_{\text{ref}}(t) + \varphi_+(t)],$
$\forall \dot{y}_t \in [\dot{y}_{\text{ref}}(t) + \varphi_-^d(t), \dot{y}_{\text{ref}}(t) + \varphi_+^d(t)], \quad \forall z_t \in Z_t$

## Feasibility assumption 2

$$\varepsilon_+ \geq \frac{(\|\varphi_-^d\| + \|\min\{\dot{\varphi}_+, 0\}\|)^2}{2\delta_-}$$
$$\varepsilon_- \geq \frac{(\|\varphi_+^d\| + \|\max\{\dot{\varphi}_-, 0\}\|)^2}{2\delta_+}$$

# Main result relative degree two

**Theorem (Bang-bang funnel controller)**

*Relative degree two + Funnels & simple switching logic + Feasibility*
$\Rightarrow$
*Bang-bang funnel controller works:*

- *existence and uniqueness of global solution*
- *error and its derivative remain within funnels for all time*
- *no zeno behaviour*

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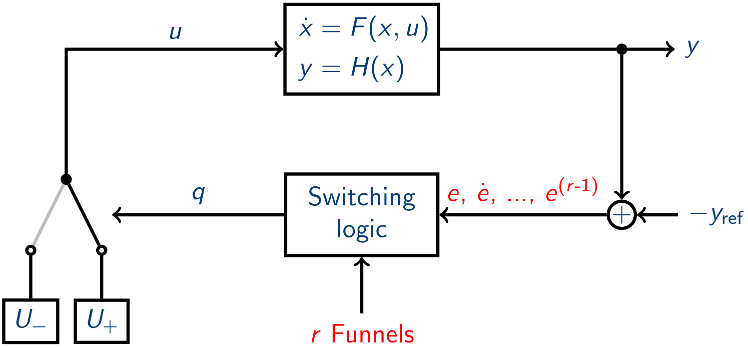
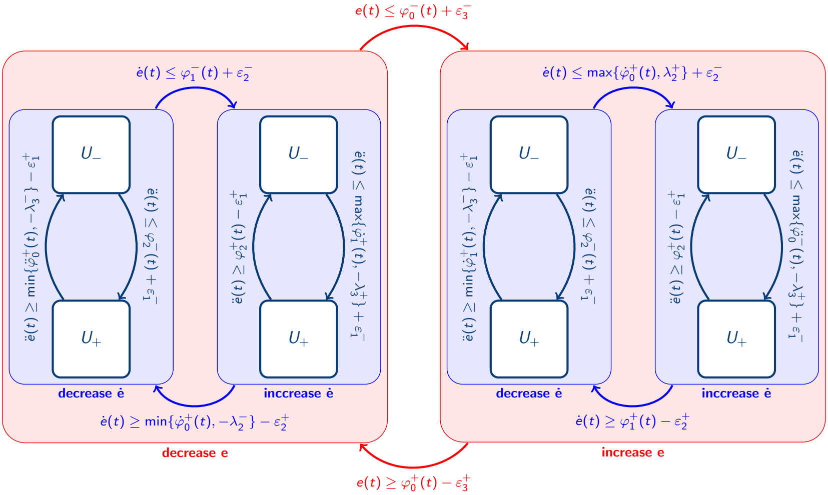
# Higher relative degree (work in progress ...)



# Switching logic



# Content

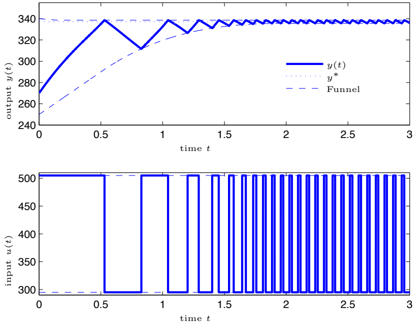
# Model of exothermic chemical reactions

Model from [Ilchmann & T. 2004]:

$$\dot{y} = br(z_1, z_2, y) - qy + u,$$
$$\dot{z}_1 = c_1 r(z_1, z_2, y) + d(z_1^{\text{in}} - z_1),$$
$$\dot{z}_2 = c_2 r(z_1, z_2, y) + d(z_2^{\text{in}} - z_2),$$

$b \geq 0,\ q > 0,\ c_1 < 0,\ c_2 \in \mathbb{R},\ d > 0,$
$z_{1/2}^{\text{in}} \geq 0$
$r : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ locally Lipschitz with $r(0,0,y) = 0\ \forall y > 0$

$y_{\text{ref}} = y^* > 0$




Feasibility assumptions from [IT 2004] imply feasibility for bang-bang funnel controller if

$$\varphi_+(t) \in (0, \overline{y} - y^*], \quad \varphi_-(t) \in (-y^*, 0),$$
$$\dot{\varphi}_+(t) > -\rho_-, \qquad \dot{\varphi}_-(t) < \rho_+,$$

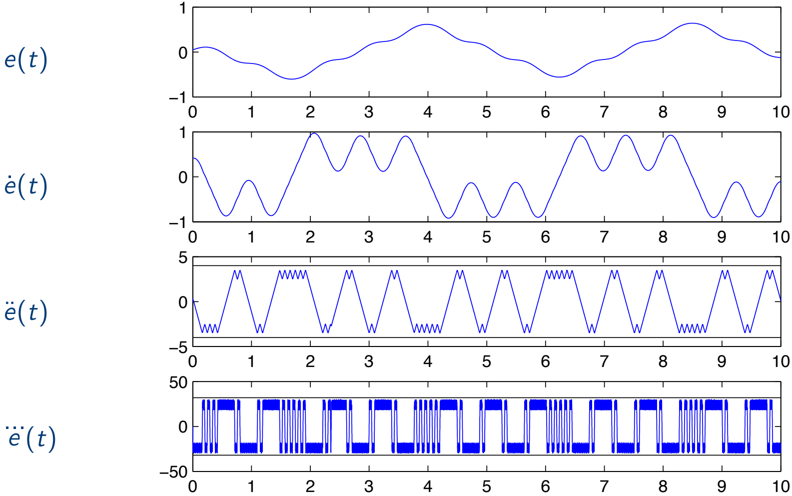
# Relative degree two experimental setup

Planned ...


**Control objective**

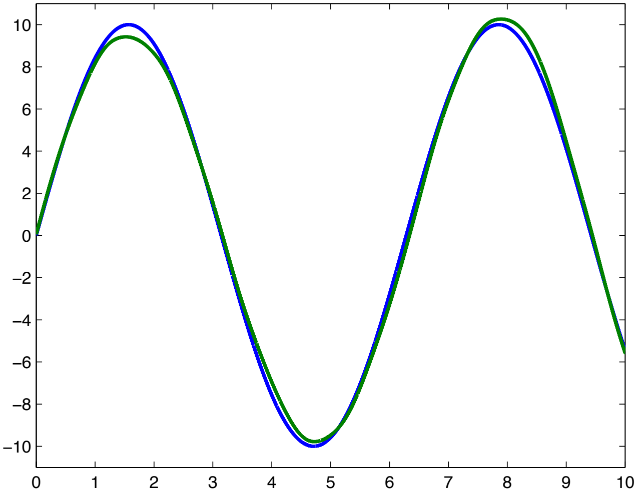
Tracking of a reference angular speed with unknown/varying load

# Relative degree 4 simulations



# Relative degree 4 simulations

$y_{\text{ref}}$: blue
$y$: green



# Conclusion

- Introduced new controller design: Bang-bang funnel controller
  - Design only depends on relative degree
  - extremely simple
- Feasibility assumptions
  - $U_+, U_-$ must be large enough
  - in terms of bounds on systems dynamics
  - higher perfomance $\Rightarrow$ larger values for $U_+, U_-$
- Switching dwell times can be guaranteed
- Higher relative degree (work in progress)
  - Switching logic remains simple (hierarchically)
  - Feasibility assumptions get more complicated
  - Switching times increase significantly (exponentially?)