Differential algebraic equations and distributional solutions

Stephan Trenn

Coordinated Science Laboratory, University of Illinois at Urbana-Champaign

Oberseminseminar Analysis
Technische Universität Dresden, 07.01.2010

ILLINOIS
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

Content

Switched DAEs

DAE = Differential algebraic equation

Homogeneous switched linear DAE

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) \qquad \textbf{(swDAE)}$$

or short $E_\sigma \dot{x} = A_\sigma x$

with

- Switching signal $\sigma : \mathbb{R} \to \{1, 2, \ldots, N\}$
  - piecewise constant, right continuous
  - locally finitely many jumps
- matrix pairs $(E_1, A_1), \ldots, (E_N, A_N)$
  - $E_p, A_p \in \mathbb{R}^{n\times n}$, $p = 1, \ldots, N$
  - (E_p, A_p) regular, i.e. $\det(E_p s - A_p) \not\equiv 0$
  - or more general: $E_p, A_p \in (\mathcal{C}^\infty)^{n\times n}$

Motivation and questions

Why switched DAEs $E_\sigma \dot{x} = A_\sigma x$?

1. Modelling electrical circuits
2. DAEs $E\dot{x} = Ax + Bu$ with switched feedback

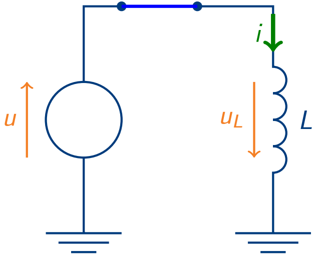
$$u(t) = F_{\sigma(t)} x(t) \quad \text{or}$$
$$u(t) = F_{\sigma(t)} x(t) + G_{\sigma(t)} \dot{x}(t)$$

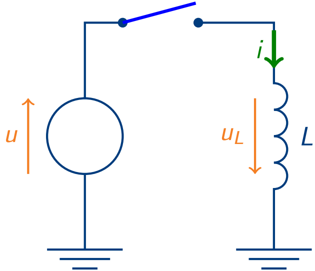
3. Approximation of time-varying DAEs $E(t)\dot{x} = A(t)x$ by piecewise-constant DAEs

Questions

1) Solution theory
2) Impulse free solutions
3) Stability

Example: Electrical circuit with coil





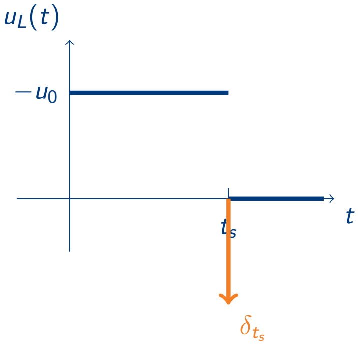
$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

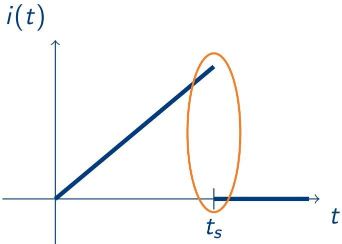
Solution of example

$\dot{u} = 0, \qquad L\frac{\mathrm{d}}{\mathrm{d}t}i = u_L, \qquad 0 = u + u_L$ or $0 = i_L$

Assume: $u(0) = u_0,\ i(0) = 0$

switch at $t_s > 0$: $\sigma(t) = \begin{cases} 1, & t < t_s \\ 2, & t \geq t_s \end{cases}$





Distribution theorie - basic ideas

Distributions - overview

- Generalized functions
- Arbitrarily often differentiable
- Dirac-Impulse δ_0 is “derivative” of jump function $\mathbb{1}_{[0,\infty)}$

Two different formal approaches

1. Functional analytical: Dual space of the space of test functions (L. Schwartz 1950)
2. Axiomatic: Space of all “derivatives” of continuous functions (J.S. Silva 1954)

Distributions - formal

Definition (Test functions)

$\mathcal{C}_0^\infty := \{\ \varphi : \mathbb{R} \to \mathbb{R} \mid \varphi \text{ is smooth with compact support } \}$

Definition (Distributions)

$\mathbb{D} := \{\ D : \mathcal{C}_0^\infty \to \mathbb{R} \mid D \text{ is linear and continuous } \}$

Definition (Regular distributions)

$f \in L_{1,\mathrm{loc}}(\mathbb{R} \to \mathbb{R})$: $\quad f_{\mathbb{D}} : \mathcal{C}_0^\infty \to \mathbb{R},\ \varphi \mapsto \int_{\mathbb{R}} f(t)\varphi(t)\mathrm{d}t \in \mathbb{D}$

Definition (Derivative)

$D'(\varphi) := -D(\varphi')$

Dirac Impulse at $t_0 \in \mathbb{R}$

$\delta_{t_0} : \mathcal{C}_0^\infty \to \mathbb{R}, \quad \varphi \mapsto \varphi(t_0)$

Multiplication with functionen

Definition (Multiplication with smooth functions)

$\alpha \in \mathcal{C}^\infty : \quad (\alpha D)(\varphi) := D(\alpha\varphi)$

(swDAE) $\quad E_\sigma \dot{x} = A_\sigma x$

Coefficients not smooth

Problem: $E_\sigma, A_\sigma \notin \mathcal{C}^\infty$

Observation:

$$\begin{array}{c} E_\sigma \dot{x} = A_\sigma x \\ i \in \mathbb{Z}: \ \sigma_{[t_i,t_{i+1})} \equiv p_i \end{array} \quad \Leftrightarrow \quad \forall i \in \mathbb{Z}: \ (E_{p_i}\dot{x})_{[t_i,t_{i+1})} = (A_{p_i}x)_{[t_i,t_{i+1})}$$

New question: Restriction of distributions

Desired properties of distributional restriction

Distributional restriction:

$$\{\ M \subseteq \mathbb{R} \mid M \text{ interval }\ \} \times \mathbb{D} \to \mathbb{D}, \quad (M, D) \mapsto D_M$$

and for each interval $M \subseteq \mathbb{R}$

1. $D \mapsto D_M$ is a projection (linear and idempotent)
2. $\forall f \in L_{1,\mathrm{loc}}: \quad (f_{\mathbb{D}})_M = (f_M)_{\mathbb{D}}$
3. $\forall \varphi \in \mathcal{C}_0^\infty: \quad \begin{bmatrix} \operatorname{supp}\varphi \subseteq M & \Rightarrow & D_M(\varphi) = D(\varphi) \\ \operatorname{supp}\varphi \cap M = \emptyset & \Rightarrow & D_M(\varphi) = 0 \end{bmatrix}$
4. $(M_i)_{i\in\mathbb{N}}$ pairwise disjoint, $M = \bigcup_{i\in\mathbb{N}} M_i$:

$$D_{M_1\cup M_2} = D_{M_1} + D_{M_2}, \quad D_M = \sum_{i\in\mathbb{N}} D_{M_i}, \quad (D_{M_1})_{M_2} = 0$$

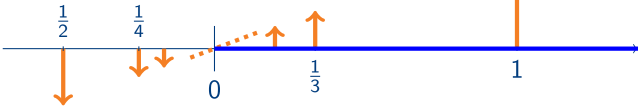
Theorem

Such a distributional restriction does not exist.

Proof of non-existence of restriction

Consider the following distribution(!):

$$D := \sum_{i\in\mathbb{N}} d_i\, \delta_{d_i}, \quad d_i := \frac{(-1)^i}{i+1}$$



Properties 2 and 3 give

$$D_{(0,\infty)} = \sum_{k\in\mathbb{N}} d_{2k}\, \delta_{d_{2k}}$$

Choose $\varphi \in \mathcal{C}_0^\infty$ such that $\varphi_{[0,1]} \equiv 1$:

$$D_{(0,\infty)}(\varphi) = \sum_{k\in\mathbb{N}} d_{2k} = \sum_{k\in\mathbb{N}} \frac{1}{2k+1} = \infty$$

Dilemma

Switched DAEs

- Examples: distributional solutions
- Multiplication with non-smooth coefficients
- Or: Restriction on intervals

Distributions

- Distributional restriction not possible
- Multiplication with non-smooth coefficients not possible
- *Initial value problems cannot be formulated*

Underlying problem

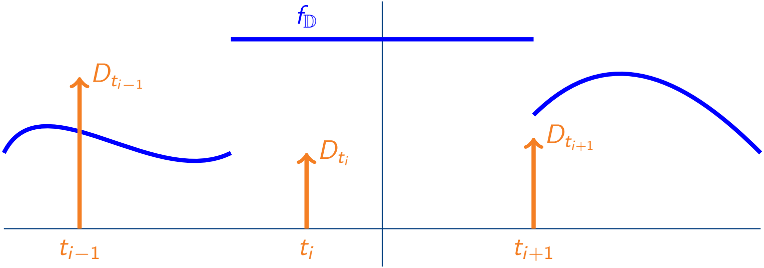
Space of distributions too big.

Piecewise smooth distributions

Define a suitable smaller space:

Definition (Piecewise smooth distributions $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$)

$$\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty} := \left\{ f_{\mathbb{D}} + \sum_{t\in T} D_t \;\middle|\; \begin{array}{l} f \in \mathcal{C}^\infty_{\mathrm{pw}}, \\ T \subseteq \mathbb{R} \text{ locally finite}, \\ \forall t \in T : D_t = \sum_{i=0}^{n_t} a_i^t \delta_t^{(i)} \end{array} \right\}$$



Properties of $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$

- $\mathcal{C}^\infty_{\text{pw}}$ "$\subseteq$" $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$
- $D \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty} \Rightarrow D' \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$
- Restriction $\mathbb{D}_{\text{pw}\mathcal{C}^\infty} \to \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$, $D \mapsto D_M$ for all intervals $M \subseteq \mathbb{R}$ well defined
- Multiplication with $\mathcal{C}^\infty_{\text{pw}}$-functions well defined
- Left and right sided evaluation at $t \in \mathbb{R}$: $D(t-), D(t+)$
- Impulse at $t \in \mathbb{R}$: $D[t]$

(swDAE) $\quad E_\sigma \dot{x} = A_\sigma x$

Application to (swDAE)

x solves (swDAE) $\quad :\Leftrightarrow \quad x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ and (swDAE) holds in $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$

Relevant questions

Consider $E_\sigma \dot{x} = A_\sigma x$ with regular matrix pairs E_p, A_p.

- Existence of solutions?
- Uniqueness of solutions?
- Inconsistent initial value problems?
- Jumps and impulses in solutions?
- Conditions for jump and impulse free solutions?

Theorem (Existence and uniqueness)

$\forall x^0 \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n \ \forall t_0 \in \mathbb{R} \ \exists! x \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$:

$$x_{(-\infty,t_0)} = {x^0}_{(-\infty,t_0)}$$
$$(E_\sigma \dot{x})_{[t_0,\infty)} = (A_\sigma x)_{[t_0,\infty)}$$

Remark: x is called *consistent solution* $\quad :\Leftrightarrow \quad E_\sigma \dot{x} = A_\sigma x$ on whole $\mathbb{R}$.

Content

Consistency projectors

For (E_i, A_i) choose S_i, T_i invertible such that (Quasi-Weierstraß form)

$$(S_i E_i T_i, S_i A_i T_i) = \left(\begin{bmatrix} I & 0 \\ 0 & N_i \end{bmatrix}, \begin{bmatrix} J_i & 0 \\ 0 & I \end{bmatrix} \right)$$

Definition (Consistency projectors)

$$\Pi_i := T_i \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_i^{-1}$$

Theorem

For all solutions x of (swDAE):

$$x(t+) = \Pi_{\sigma(t)} x(t-)$$

Impulse and jump freeness

Theorem (Impulse freeness)

If for (swDAE)

$$\forall p, q \in \{1, \dots, N\} : E_p(I - \Pi_p)\Pi_q = 0,$$

then all consistent solutions $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})$ are impulse free.

Basic idea of proof:
$x(t+) - x(t-) \in \operatorname{im}(I - \Pi_p)\Pi_q$ and $E_p\dot{x}[t] = 0 \quad \Rightarrow \quad x[t] = 0.$

Theorem (Jump freeness)

If for (swDAE)

$$\forall p, q \in \{1, \dots, N\} : (I - \Pi_p)\Pi_q = 0,$$

then all consistent solutions $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})$ are jump and impulse free.

Examples revisited

$$(E_1, A_1) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \right) \quad \Rightarrow \quad \Pi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$(E_2, A_2) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right) \quad \Rightarrow \quad \Pi_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Jumps? $(I - \Pi_1)\Pi_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad (I - \Pi_1)\Pi_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

Impulses? $E_1(I - \Pi_1)\Pi_2 = 0, \quad E_2(I - \Pi_2)\Pi_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Conclusion and outlook

Conclusion:

- Motivation for switched DAEs
- Distributional solution: Needed, but impossible
- Solution: Piecewise-smooth distributions
- Applications of solution theory: Conditions for impulse freeness of solutions

Outlook and further results

- Multiplication defined for $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$, e.g. ${\delta_t}^2 = 0$
- DAEs $E\dot{x} = Ax + f$ with distributional coefficients can be studied, e.g. $\dot{x} = \delta_0 x$
- Stability results