On stability of switched DAEs

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I L L I N O I S

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Contents

Switched DAEs

DAE = Differential algebraic equation

Homogeneous switched linear DAE

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) \qquad \textbf{(swDAE)}$$

or short $E_\sigma \dot{x} = A_\sigma x$

with

- switching signal $\sigma : \mathbb{R} \to \{1, 2, \ldots, N\}$
  - piecewise constant
  - locally finite jumps
- matrix pairs $(E_1, A_1), \ldots, (E_N, A_N)$
  - $E_p, A_p \in \mathbb{R}^{n\times n}$, $p = 1, \ldots, N$
  - (E_p, A_p) regular, i.e. $\det(E_p s - A_p) \not\equiv 0$

Questions

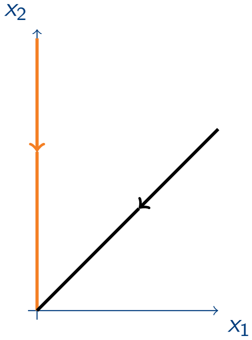
Existence and nature of solutions?

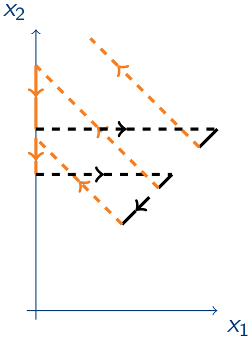
$E_p\dot{x} = A_p x$ asymp. stable $\forall p \quad \overset{?}{\Rightarrow} \quad E_\sigma\dot{x} = A_\sigma x$ asymp. stable

Example 1

Example 1:

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \right) \quad (E_2, A_2) = \left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \right)$$

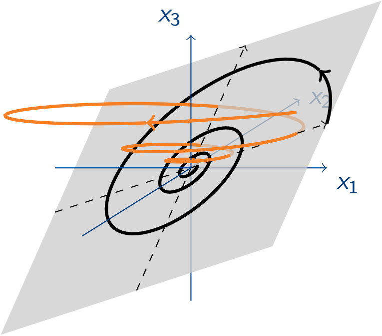


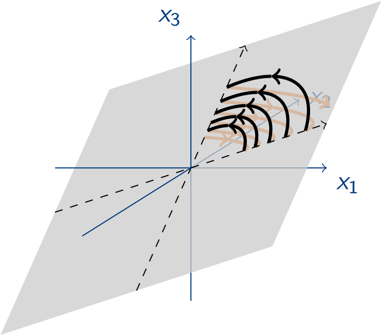


Example 2

Example 2:

$$(E_1, A_1) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 8\pi & 0 \\ \pi/2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right) \qquad (E_2, A_2) = \left(\begin{bmatrix} 0 & 4 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -4\pi & -4 & 0 \\ -1 & 4\pi & 0 \\ -1 & -4 & 4 \end{bmatrix} \right)$$





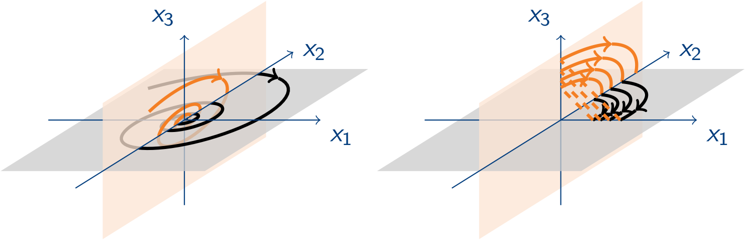
Switching signal:

$\Delta t = 1/4$

Example 3

Example 3:

$$(E_1, A_1) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 2\pi & 0 \\ -2\pi & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \quad (E_2, A_2) = \left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 4\pi & -1 & 4\pi \\ -1 & \pi & -1 \\ 1 & 0 & 0 \end{bmatrix} \right)$$



Switching signal:

$\Delta t = 1/4$

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Solutions of classical DAEs

Consider for now non-switched DAE

$$E\dot{x} = Ax.$$

Theorem (Weierstrass 1868)

(E, A) regular $\Leftrightarrow$ $\exists S, T \in \mathbb{R}^{n\times n}$ invertible:

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right),$$

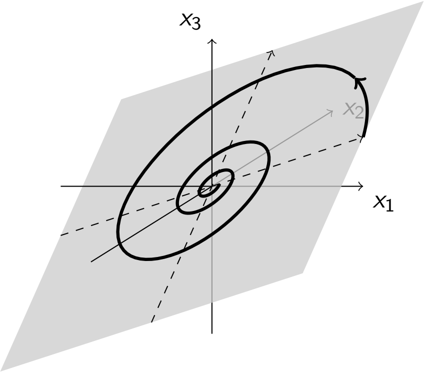
N nilpotent

Corollary (for regular (E, A))

x solves $E\dot{x} = Ax \Leftrightarrow x(t) = T \begin{pmatrix} e^{Jt} v_0 \\ 0 \end{pmatrix}$

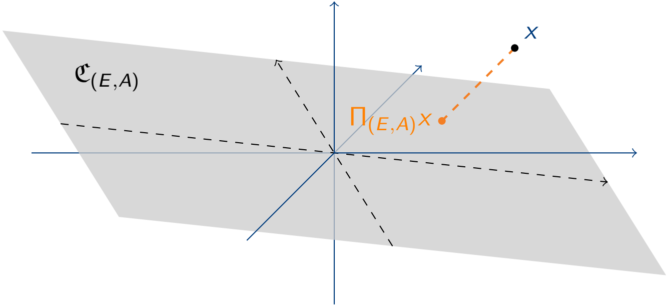
Consistency space: $\mathfrak{C}_{(E,A)} := T \begin{pmatrix} * \\ 0 \end{pmatrix}$

$$(E, A) = \left(\begin{bmatrix} 0 & 4 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -4\pi & -4 & 0 \\ -1 & 4\pi & 0 \\ -1 & -4 & 4 \end{bmatrix} \right)$$



$$T = \begin{bmatrix} 0 & 4 & * \\ 1 & 0 & * \\ 1 & 1 & * \end{bmatrix}, \quad J = \begin{bmatrix} -1 & -4\pi \\ \pi & -1 \end{bmatrix}$$

Consistency projectors



Definition (Consistency projectors for regular (E,A))

Let $S, T \in \mathbb{R}^{n\times n}$ be invertible with $(SET, SAT) = \left(\left[\begin{smallmatrix} I & 0 \\ 0 & N \end{smallmatrix}\right], \left[\begin{smallmatrix} J & 0 \\ 0 & I \end{smallmatrix}\right]\right)$:

$$\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$$

Lyapunov functions for regular (E, A)

Definition (Lyapunov function for $E\dot{x} = Ax$)

$Q = \overline{Q}^\top > 0$ on $\mathfrak{C}_{(E,A)}$ and $P = \overline{P}^\top > 0$ solves

$$A^\top PE + E^\top PA = -Q \quad \text{(generalized Lyapunov equation)}$$

Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0} : x \mapsto (Ex)^\top PEx$

Theorem (Owens & Debeljkovic 1985)

$E\dot{x} = Ax$ asymptotically stable $\quad\Leftrightarrow\quad$ *$\exists$ Lyapunov function*

Remark (Other definitions for Lyapunov functions)

Other definition for Lyapunov functions are possible, for example

$$V(x) = (Ex)^\top Px$$

where (E, A) is index one and $A^\top P + P^\top A = -I$, $P^\top E = E^\top P \geq 0$.

Intermediate summary: Problems and their solutions

Consider again switched DAE

$$E_\sigma \dot{x} = A_\sigma x \qquad \textbf{(swDAE)}$$

1. Stability criteria for single DAEs $E_p \dot{x} = A_p x$
   $\Rightarrow$ Lyapunov functions
2. **No classical solutions for switched DAEs**
   $\Rightarrow$ Allow for jumps in solutions
3. How does inconsistent initial value "jump" to consistent one?
   $\Rightarrow$ Consistency projectors $\Pi_{(E_1,A_1)}, \ldots, \Pi_{(E_N,A_N)}$
4. Differentiation of jumps
   $\Rightarrow$ Space of Distributions as solution space
5. **Multiplication with non-smooth coefficients**
   $\Rightarrow$ Space of piecewise-smooth distributions
   $\Rightarrow$ Existence and uniqueness of solutions

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Asymptotic stability and impulse free solutions

$$E_\sigma \dot{x} = A_\sigma x \qquad \textbf{(swDAE)}$$

Definition (Asymptotic stability of switched DAE)

(swDAE) asymptotically stable $:\Leftrightarrow$
$\forall$ distr. solutions x: $\lim_{t\to\infty} x(t) = 0$ and x is impulse free

Let $\Pi_p := \Pi_{(E_p,A_p)}$ be the consistency projectors of (E_p, A_p)

Impulse freeness condition

(IFC): $\forall p,q \in \{1,\ldots,N\} : E_p(I-\Pi_p)\Pi_q = 0$

Theorem (T. 2009)

(IFC) $\Rightarrow$ *All solutions of* $E_\sigma \dot{x} = A_\sigma x$ *are impulse free*

Stability under arbitrary switching

Consider **(swDAE)** with additional assumption:

($\exists$V$_p$): $\forall p \in \{1, \ldots, N\}$ $\exists$ Lyapunov function V_p for (E_p, A_p)

i.e. each DAE (E_p, A_p) is asymp. stable

Lyapunov jump condition

(LJC): $\forall p, q = 1, \ldots, N \ \forall x \in \mathfrak{C}_{(E_q, A_q)}: \quad V_p(\Pi_p x) \leq V_q(x)$

Theorem (Liberzon and T. 2009)

(IFC) $\wedge$ **($\exists$V$_p$)** $\wedge$ **(LJC)** $\Rightarrow$ **(swDAE)** *asymptotically stable*

Examples 1, 2 and 3 all satisfy **(IFC)** and **($\exists$V$_p$)**,
but none fulfills **(LJC)**

Jump free switching

Consider special case, where switching does not induce jumps. For $x^0 \in \mathbb{R}^n$ define

$$\Sigma_{x^0} := \left\{ \sigma : \mathbb{R} \to \{1, \ldots, N\} \;\middle|\; \begin{array}{l} \exists \text{ solution } x \text{ of } \textbf{(swDAE)} \\ \text{with } x(0) = x^0 \text{ and} \\ x \text{ has no jumps} \end{array} \right\}$$

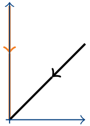
Weak Lyapunov condition

(wLC): $\forall p, q = 1, \ldots, N \ \forall x \in \mathfrak{C}_{(E_p,A_p)} \cap \mathfrak{C}_{(E_q,A_q)} : \quad V_p(x) = V_q(x)$

Theorem (Liberzon & T. 2009)

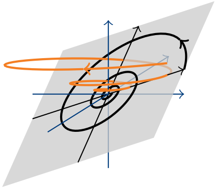
$\sigma \in \Sigma_{x^0} \wedge x$ *solution of* **(swDAE)** *with* $x(0) = x^0 \wedge (\exists \mathbf{V_p}) \wedge$ **(wLC)**
$\Rightarrow \quad x(t) \to 0$ *and* x *impulse free*

Examples revisited



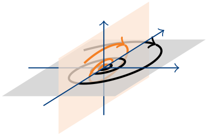
Example 1:

$(\exists \mathbf{V_p})$ and **(wLC)** fulfilled
BUT: Σ_{x^0} is "empty" when $x^0 \neq 0$
i.e.: result not useful here



Example 2:

$(\exists \mathbf{V_p})$ and $\Sigma_{x^0} \neq \emptyset$ for some x^0
BUT: **(wLC)** not satisfied



Example 3:

All conditions fulfilled!
$\Rightarrow$ all jump free solutions converge

Slow switching

Slow switching signals with dwell time $\tau_d > 0$:

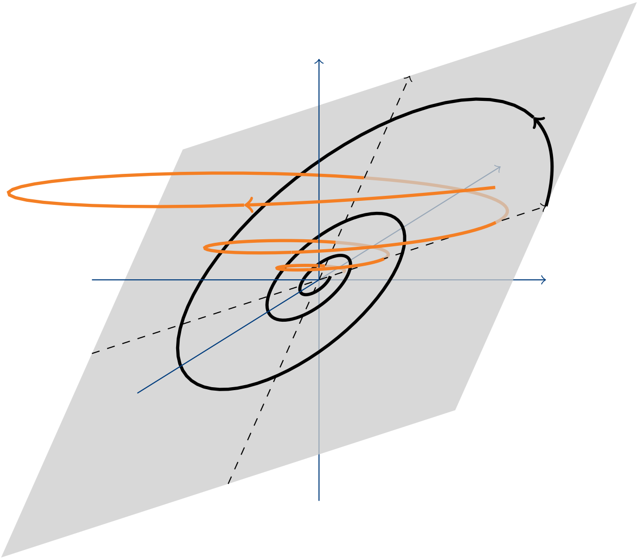
$$\Sigma^{\tau_d} := \left\{ \sigma : \mathbb{R} \to \{1, \ldots, N\} \;\middle|\; \begin{array}{l} \forall \text{ switching times} \\ t_i \in \mathbb{R}, i \in \mathbb{Z} : \\ t_{i+1} - t_i \geq \tau \end{array} \right\}.$$

Theorem (Liberzon & T. 2009)

$\exists \tau_d > 0 \; \forall \sigma \in \Sigma^{\tau_d}$: **(IFC)** $\wedge$ **($\exists$V$_p$)** $\Rightarrow$ **(swDAE)** *asymptotically stable*

As a reminder:
(IFC): $\forall p, q \in \{1, \ldots, N\} : E_p(I - \Pi_{(E_p,A_p)})\Pi_{(E_q,A_q)} = 0$

Thanks for your attention!



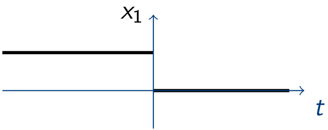
Distributional solutions

Example (Inconsistent initial values)

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \left(\Leftrightarrow \begin{array}{l} x_1 = 0 \\ x_2 = \dot{x}_1 \end{array} \right) \quad \text{on } [0, \infty)$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{on } (-\infty, 0)$$

Obviously: $x_1 = \mathbb{1}_{(-\infty,0)}$



$$x_2 = \begin{cases} 0, & \text{auf } (-\infty, 0) \\ \dot{x}_1 = -\delta_0, & \text{auf } [0, \infty) \end{cases}$$

hence: $x_2 = -\delta_0$ (Dirac impulse)

Existence and uniqueness of solutions

In the following: Space of piecewise smooth distributions distributions as solution space.

Consider $E_\sigma \dot{x} = A_\sigma x$ with

- $\sigma : \mathbb{R} \to \{1, \ldots, N\}$, locally finite jumps
- $(E_1, A_1), \ldots, (E_N, A_N)$ regular

Theorem (T. 2009)

For each initial trajectory $x^0 : (-\infty, 0) \to R^n$ exists a unique distributional solution of

$$\begin{aligned} x &= x^0 && \text{on } (-\infty, 0) \\ E_\sigma \dot{x} &= A_\sigma x && \text{on } [0, \infty) \end{aligned}$$

Remark:

x distr. solution of $E_\sigma \dot{x} = A_\sigma x$

$$\Rightarrow \quad \forall t \in \mathbb{R} : x(t+) = \Pi_{(E_{\sigma(t)}, A_{\sigma(t)})} x(t-)$$

Calculation of Consistency projectors

Theorem (Quasi-Weierstraß form, Berger, Ilchmann, T. 2009)

Let (E, A) be regular.

$$\mathcal{V}_0 := \mathbb{R}^n, \qquad \mathcal{V}_{k+1} := A^{-1}(E\mathcal{V}_k),\ k = 0, 1, \ldots, k^*$$
$$\mathcal{W}_0 := \{0\}, \qquad \mathcal{W}_{k+1} := E^{-1}(A\mathcal{W}_k),\ k = 0, 1, \ldots, k^*.$$

Let $\operatorname{im} V = \mathcal{V}_{k^*}$, $\operatorname{im} W = \mathcal{W}_{k^*}$ *and* $T := [V, W]$, $S^{-1} := [EV, AW]$ *then*

$$(SET, SAT) = \left(\begin{bmatrix} I & \\ & N \end{bmatrix}, \begin{bmatrix} J & \\ & I \end{bmatrix} \right).$$

Remark:

- $\mathcal{V}_k \supseteq \mathcal{V}_{k+1}$ and $\mathcal{W}_k \subseteq \mathcal{W}_{k+1}$
- V and W are easily computable (e.g. with a Matlab)
- Hence $\Pi_{(E,A)} = T \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] T^{-1}$ easily computable

Matlab Code for calculating the consistency projectors

Calculating a basis of the pre-image $A^{-1}(\operatorname{im} S)$:

```
function V=getPreImage(A,S)
[m1,n1]=size(A); [m2,n2]=size(S);
if m1==m2 | m2==0
    H=null([A,S]);
    V=colspace(H(1:n1,:));
end;
```

Calculating V with $\operatorname{im} V = \mathcal{V}_{k^*}$:

```
function V = getVspace(E,A)
[m,n]=size(E);
if (m==n) & size(E)==size(A)
    V=eye(n,n);
    oldsize=n; newsize=n; finished=0;
    while finished==0;
       EV=colspace(E*V);
       V=getPreImage(A,EV);
       oldsize=newsize;
       newsize=rank(V);
       finished = (newsize==oldsize);
    end;
end;
```

Calculating W with $\operatorname{im} W = \mathcal{W}_{k^*}$ analog.