

A solution theory for switched differential algebraic equations

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Content



- 1 Introduction
 - System class
- 2 Distributions as solutions
- 3 Solution theory for switched DAEs
- 4 Impulse and jump freeness of solutions

Switched DAEs



Homogeneous switched linear DAE (differential algebraic equation):

$$\text{(swDAE)} \quad E_{\sigma(t)} \dot{x}(t) = A_{\sigma(t)} x(t) \quad \text{or} \quad E_{\sigma} \dot{x} = A_{\sigma} x$$

with

- Switching signal $\sigma : \mathbb{R} \rightarrow \{1, 2, \dots, N\}$
 - piecewise constant, right continuous
 - locally finitely many jumps
- matrix pairs $(E_1, A_1), \dots, (E_N, A_N)$
 - $E_p, A_p \in \mathbb{R}^{n \times n}$, $p = 1, \dots, N$
 - (E_p, A_p) regular, i.e. $\det(E_p s - A_p) \neq 0$

Motivation and questions



Why switched DAEs $E_{\sigma}\dot{x} = A_{\sigma}x$?

- 1 Modelling electrical circuits
- 2 DAEs $E\dot{x} = Ax + Bu$ with switched feedback

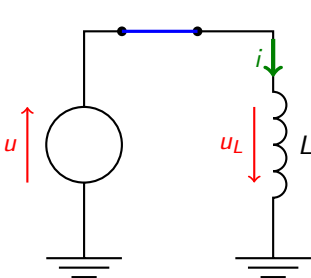
$$u(t) = F_{\sigma(t)}x(t) \quad \text{oder}$$
$$u(t) = F_{\sigma(t)}x(t) + G_{\sigma(t)}\dot{x}(t)$$

- 3 Approximation of time-varying DAEs $E(t)\dot{x} = A(t)x$ by piecewise-constant DAEs

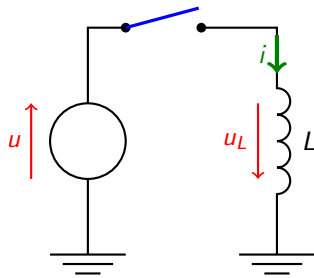
Questions

- 1) Solution theory
- 2) Impulse free solutions
- 3) Stability

Example: Electrical circuit with coil



$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$



$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

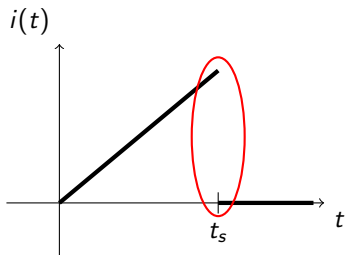
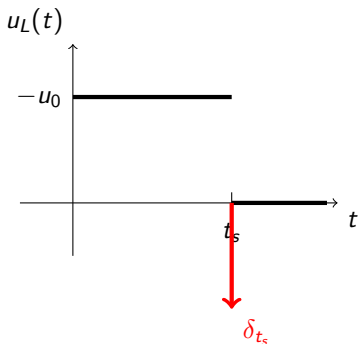
Solution of example

Or

$$\dot{u} = 0, \quad L \frac{d}{dt} i = u_L, \quad 0 = u + u_L \text{ or } 0 = i_L$$

Assume: $u(0) = u_0, i(0) = 0$

$$\text{switch at } t_s > 0: \sigma(t) = \begin{cases} 1, & t < t_s \\ 2, & t \geq t_s \end{cases}$$



Distribution theorie - basic ideas



Distributions - overview

- Generalized functions
- Arbitrarily often differentiable
- Dirac-Impulse δ_0 is “derivative” of jump function $\mathbb{1}_{[0,\infty)}$

Two different formal approaches

- 1 Functional analytical: Dual space of the space of test functions (L. Schwartz 1950)
- 2 Axiomatic: Space of all “derivatives” of continuous functions (J.S. Silva 1954)

Distributions - formal



Definition (Test functions)

$$\mathcal{C}_0^\infty := \{ \varphi : \mathbb{R} \rightarrow \mathbb{R} \mid \varphi \text{ is smooth with compact support} \}$$

Definition (Distributions)

$$\mathbb{D} := \{ D : \mathcal{C}_0^\infty \rightarrow \mathbb{R} \mid D \text{ is linear and continuous} \}$$

Definition (Regular distributions)

$$f \in L_{1,\text{loc}}(\mathbb{R} \rightarrow \mathbb{R}): \quad f_{\mathbb{D}} : \mathcal{C}_0^\infty \rightarrow \mathbb{R}, \quad \varphi \mapsto \int_{\mathbb{R}} f(t)\varphi(t)dt \in \mathbb{D}$$

Definition (Derivative)

$$D'(\varphi) := -D(\varphi')$$

Dirac Impulse at $t_0 \in \mathbb{R}$

$$\delta_{t_0} : \mathcal{C}_0^\infty \rightarrow \mathbb{R}, \quad \varphi \mapsto \varphi(t_0)$$

Multiplication with functionen

Definition (Multiplication with smooth functions)

$$\alpha \in \mathcal{C}^\infty : (\alpha D)(\varphi) := D(\alpha\varphi)$$

$$(\text{swDAE}) \quad E_\sigma \dot{x} = A_\sigma x$$

Coefficients not smooth

Problem: $E_\sigma, A_\sigma \notin \mathcal{C}^\infty$

Observation:

$$E_\sigma \dot{x} = A_\sigma x \quad \Leftrightarrow \quad \forall i \in \mathbb{Z} : (E_{p_i} \dot{x})_{[t_i, t_{i+1})} = (A_{p_i} x)_{[t_i, t_{i+1})}$$
$$i \in \mathbb{Z} : \sigma_{[t_i, t_{i+1})} \equiv p_i$$

New question: **Restriction of distributions**

Desired properties of distributional restriction

Distributional restriction:

$$\{ M \subseteq \mathbb{R} \mid M \text{ interval} \} \times \mathbb{D} \rightarrow \mathbb{D}, \quad (M, D) \mapsto D_M$$

and for each interval $M \subseteq \mathbb{R}$

- ① $D \mapsto D_M$ is a projection (linear and idempotent)
- ② $\forall f \in L_{1,loc} : (f_{\mathbb{D}})_M = (f_M)_{\mathbb{D}}$
- ③ $\forall \varphi \in C_0^\infty : \left[\begin{array}{ll} \text{supp } \varphi \subseteq M & \Rightarrow D_M(\varphi) = D(\varphi) \\ \text{supp } \varphi \cap M = \emptyset & \Rightarrow D_M(\varphi) = 0 \end{array} \right]$
- ④ $(M_i)_{i \in \mathbb{N}}$ pairwise disjoint, $M = \bigcup_{i \in \mathbb{N}} M_i$:

$$D_{M_1 \cup M_2} = D_{M_1} + D_{M_2}, \quad D_M = \sum_{i \in \mathbb{N}} D_{M_i}, \quad (D_{M_1})_{M_2} = 0$$

Theorem

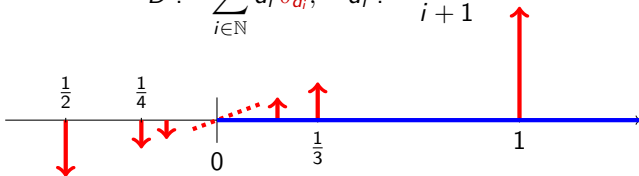
Such a distributional restriction does not exist.

Proof of non-existence of restriction



Consider the following distribution(!):

$$D := \sum_{i \in \mathbb{N}} d_i \delta_{d_i}, \quad d_i := \frac{(-1)^i}{i+1}$$



Properties 2 and 3 give

$$D_{(0,\infty)} = \sum_{k \in \mathbb{N}} d_{2k} \delta_{d_{2k}}$$

Choose $\varphi \in \mathcal{C}_0^\infty$ such that $\varphi_{[0,1]} \equiv 1$:

$$D_{(0,\infty)}(\varphi) = \sum_{k \in \mathbb{N}} d_{2k} = \sum_{k \in \mathbb{N}} \frac{1}{2k+1} = \infty$$

Dilemma



Switched DAEs

- Examples: distributional solutions
- Multiplication with non-smooth coefficients
- Or: Restriction on intervals

Distributions

- Distributional restriction not possible
- Multiplication with non-smooth coefficients not possible
- *Initial value problems cannot be formulated*

Underlying problem

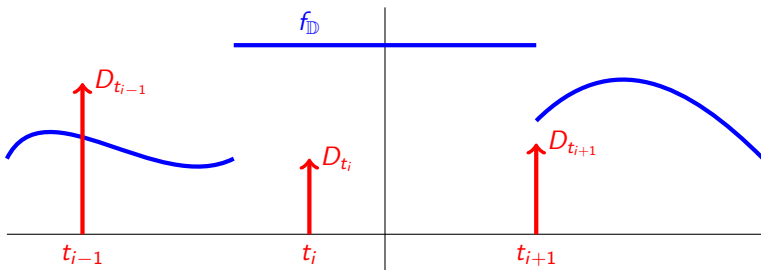
Space of distributions **too big**.

Piecewise smooth distributions

Define a suitable smaller space:

Definition (Piecewise smooth distributions $\mathbb{D}_{\text{pw}C^\infty}$)

$$\mathbb{D}_{\text{pw}C^\infty} := \left\{ f_{\mathbb{D}} + \sum_{t \in T} D_t \mid \begin{array}{l} f \in C_{\text{pw}}^\infty, \\ T \subseteq \mathbb{R} \text{ locally finite,} \\ \forall t \in T : D_t = \sum_{i=0}^{n_t} a_i^t \delta_t^{(i)} \end{array} \right\}$$



Properties of $\mathbb{D}_{pw}\mathcal{C}^\infty$

- \mathcal{C}_{pw}^∞ “ \subseteq ” $\mathbb{D}_{pw}\mathcal{C}^\infty$
- $D \in \mathbb{D}_{pw}\mathcal{C}^\infty \Rightarrow D' \in \mathbb{D}_{pw}\mathcal{C}^\infty$
- Restriction $\mathbb{D}_{pw}\mathcal{C}^\infty \rightarrow \mathbb{D}_{pw}\mathcal{C}^\infty$, $D \mapsto D_M$ for all intervals $M \subseteq \mathbb{R}$ well defined
- Multiplication with \mathcal{C}_{pw}^∞ -functions well defined
- Left and right sided evaluation at $t \in \mathbb{R}$: $D(t-), D(t+)$
- Impulse at $t \in \mathbb{R}$: $D[t]$

(swDAE) $E_\sigma \dot{x} = A_\sigma x$

Application to (swDAE)

x solves (swDAE) $:\Leftrightarrow x \in (\mathbb{D}_{pw}\mathcal{C}^\infty)^n$ and (swDAE) holds in $\mathbb{D}_{pw}\mathcal{C}^\infty$

Relevant questions



Consider $E_\sigma \dot{x} = A_\sigma x$.

- Existence of solutions?
- Uniqueness of solutions?
- Inconsistent initial value problems?
- Jumps and impulses in solutions?
- Conditions for jump and impulse free solutions?

Theorem (Existence and uniqueness)

$\forall x^0 \in (\mathbb{D}_{\text{pwC}^\infty})^n \quad \forall t_0 \in \mathbb{R} \quad \exists! x \in (\mathbb{D}_{\text{pwC}^\infty})^n$:

$$\begin{aligned}x_{(-\infty, t_0)} &= x^0_{(-\infty, t_0)} \\(E_\sigma \dot{x})_{[t_0, \infty)} &= (A_\sigma x)_{[t_0, \infty)}\end{aligned}$$

Remark: x is called *consistent solution* $\Leftrightarrow E_\sigma \dot{x} = A_\sigma x$ on whole \mathbb{R} .

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Consistency projectors



For (E_i, A_i) choose S_i, T_i invertible such that

$$(S_i E_i T_i, S_i A_i T_i) = \left(\begin{bmatrix} I & 0 \\ 0 & N_i \end{bmatrix}, \begin{bmatrix} J_i & 0 \\ 0 & I \end{bmatrix} \right)$$

Definition (Consistency projectors)

$$\Pi_i := T_i \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_i^{-1}$$

Theorem

For all solutions x of (swDAE):

$$x(t+) = \Pi_{\sigma(t)} x(t-)$$

Impulse and jump freeness

Theorem (Impulse freeness)

If for (swDAE)

$$\forall p, q \in \{1, \dots, N\} : E_p(I - \Pi_p)\Pi_q = 0,$$

then all consistent solutions $x \in (\mathbb{D}_{\text{pw}}\mathcal{C}^\infty)$ are impulse free.

Basic idea of proof:

$$x(t+) - x(t-) \in \text{im}(I - \Pi_p)\Pi_q \text{ and } E_p\dot{x}[t] = 0 \Rightarrow x[t] = 0.$$

Theorem (Jump freeness)

If for (swDAE)

$$\forall p, q \in \{1, \dots, N\} : (I - \Pi_p)\Pi_q = 0,$$

then all consistent solutions $x \in (\mathbb{D}_{\text{pw}}\mathcal{C}^\infty)$ are jump and impulse free.

Examples revisited



$$(E_1, A_1) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \right) \Rightarrow \Pi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$(E_2, A_2) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right) \Rightarrow \Pi_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Jumps? } (I - \Pi_1)\Pi_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad (I - \Pi_2)\Pi_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\text{Impulses? } E_1(I - \Pi_1)\Pi_2 = 0, \quad E_2(I - \Pi_2)\Pi_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Conclusion and outlook



Conclusion:

- Motivation for switched DAEs
- Distributional solution: Needed, but impossible
- Solution: Piecewise-smooth distributions
- Applications of solution theory: Conditions for impulse freeness of solutions

Outlook and further results

- Multiplication defined for $\mathbb{D}_{\text{pw}C^\infty}$, e.g. $\delta_t^2 = 0$
- DAEs $E\dot{x} = Ax + f$ with distributional coefficients can be studied, e.g. $\dot{x} = \delta_0 x$
- Stability results