Linear differential-algebraic equations with piecewise smooth coefficients

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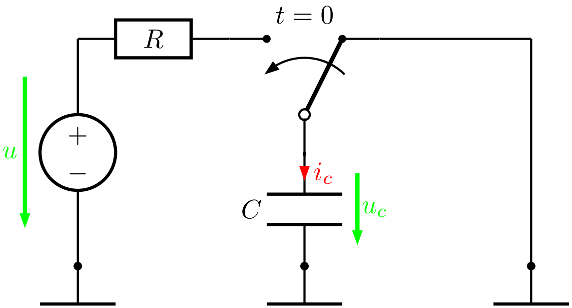
Content

A simple example



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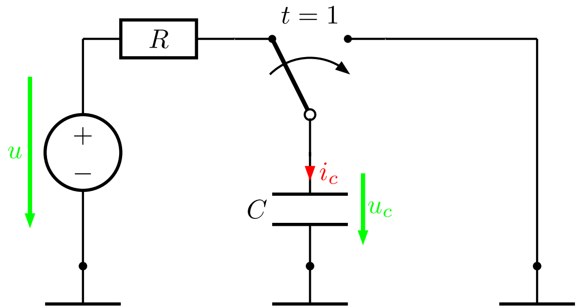

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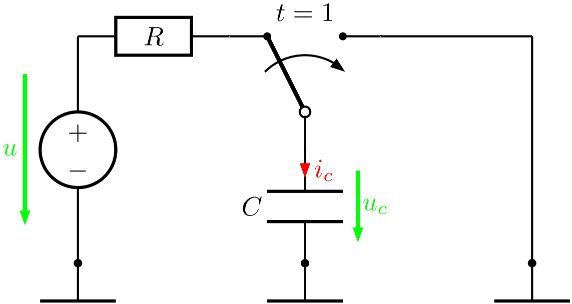


Capacitor equation: $C\frac{\mathrm{d}}{\mathrm{d}t}u_c(t) = i_c(t),\ t \in \mathbb{R}$

Kirchhoff's law: $u_c(t) = \begin{cases} u(t) - Ri_c(t), & t \in [0,1) \\ 0, & \text{otherwise} \end{cases}$

Linear time-varying DAE

Definition (Linear time-varying DAE)

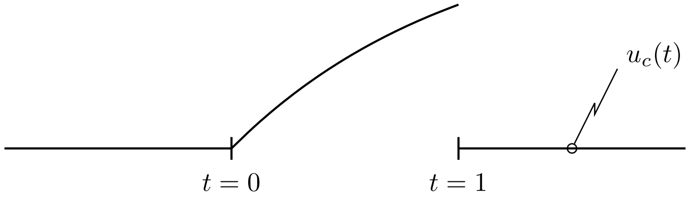
$$E(\cdot)\dot{x} = A(\cdot)x + f$$

Example: $x_1 = u_c$, $x_2 = i_c$

$$E(t) = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}, \quad A(t) = \begin{cases} \begin{bmatrix} 0 & 1 \\ 1 & R \end{bmatrix}, & t \in [0,1) \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & \text{otherwise} \end{cases}$$

$$f(t) = \begin{cases} u(t), & t \in [0,1) \\ 0, & \text{otherwise} \end{cases}$$

Solution of example



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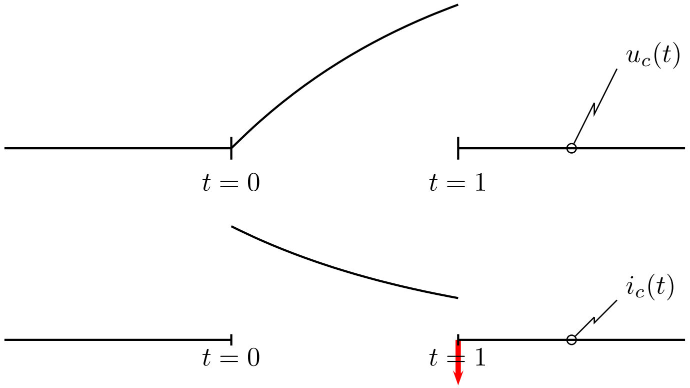

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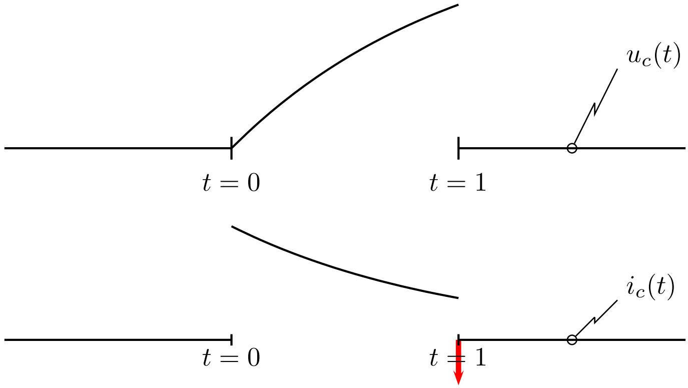

Conclusion

Solution theory of DAEs needs distributional solutions.

Content

Basic properties of distributions

Distributions - informal

- Generalized functions
- Arbitrarily often differentiable

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- Arbitrarily often differentiable

Definition (Test functions)

$\mathbf{\Phi} := \{ \ \varphi : \mathbb{R} \to \mathbb{R} \ | \ \varphi \text{ is smooth with bounded support } \}$

Definition (Distributions)

$\mathbb{D} := \{ \ D : \Phi \to \mathbb{R} \ | \ D \text{ is linear und continuous } \} = \Phi'$

Definition (Support of distribution)

$\text{supp} D := \left(\bigcup \{ \ M \subseteq \mathbb{R} \ | \ \forall \varphi \in \Phi : \text{supp} \varphi \subseteq M \Rightarrow D(\varphi) = 0 \ \} \right)^C$

Basic properties of distributions

Definition (Regular distributions)

$f \in L_{1,\mathrm{loc}}(\mathbb{R} \to \mathbb{R})$: $\quad f_{\mathbb{D}} : \Phi \to \mathbb{R},\ \varphi \mapsto \int_{\mathbb{R}} \varphi(t) f(t) \mathrm{d}t$

Dirac impulse at $t \in \mathbb{R}$

$\delta_t : \Phi \to \mathbb{R}, \quad \varphi \mapsto \varphi(t)$

Definition (Derivative of distributions)

$D'(\varphi) := -D(\varphi')$

Definition (Multiplication with smooth function $a : \mathbb{R} \to \mathbb{R}$)

$(aD)(\varphi) := D(\mathbf{a}\boldsymbol{\varphi})$

Distributional DAEs

Definition (Distributional DAE)

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Problem

Only well defined if E and A are constant or smooth!

$\Rightarrow$ Multiplication aD for non-smooth $a : \mathbb{R} \to \mathbb{R}$ must be studied.

Content

Multiplication with non-smooth functions

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Definition (Piecewise W^n distributions)

$D \in \mathbb{D}_{\mathrm{pw}\mathbf{W}^n} :\Leftrightarrow D = f_{\mathbb{D}} + \sum_i D_i$

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- $D_i \in \mathbb{D}$, $i \in \mathbb{Z}$, are distributions with point support $\{t_i\}$
- the support of all D_i has no accumulation points

Properties of piecewise W^n distributions

Piecewise regular distributions

$W^n{}_{\text{pw}}(\mathbb{R} \to \mathbb{R}) \subseteq W^0{}_{\text{pw}}(\mathbb{R} \to \mathbb{R}) = L_{1,\text{loc}}(\mathbb{R} \to \mathbb{R})$

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$D \in \mathbb{D}_{\text{pw}W^{n+1}} \quad \Rightarrow \quad D' \in \mathbb{D}_{\text{pw}W^n}.$

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Lemma

$D \in \mathbb{D}_{\mathrm{pw}W^{n+1}} \quad \Rightarrow \quad D' \in \mathbb{D}_{\mathrm{pw}W^n}.$

Definition (Restriction of piecewise regular distributions)

$D = f_{\mathbb{D}} + \sum_i D_i \in \mathbb{D}_{\mathrm{pw}},\ M \subseteq \mathbb{R}$

$$D_M := (f_M)_{\mathbb{D}} + \sum_i \mathbb{1}_M(t_i) D_i$$

Multiplication with non-smooth functions

Definition (Piecewise smooth functions)

$a \in \mathcal{C}^\infty{}_{\text{pw}}(\mathbb{R} \to \mathbb{R}) \quad :\Leftrightarrow \quad a = \sum_j \mathbb{1}_{I_j} a_j,$
where $a_j \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R})$ and $I_j = [t_j, t_{j+1})$ for $j \in \mathbb{Z}$.

Note: Representation is not unique!

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Definition (Multiplication with piecewise smooth functions)

$D \in \mathbb{D}_{\text{pw}}$, $a \in \mathcal{C}^\infty{}_{\text{pw}}(\mathbb{R} \to \mathbb{R})$

$$aD := \sum_j a_j D_{I_j}$$

Properties of multiplication with non-smooth functions

Properties

- aD does not depend on the specific representation of a
- aD is again a distribution, i.e. linear and continuous
- aD "behaves" like multiplication, e.g.
  $(a_1 + a_2)D = a_1D + a_2D$, ...
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$\Rightarrow$ Distributional DAE

$$E(\cdot)X' = A(\cdot)X + F$$

with piecewise smooth coefficients makes sense!

Content

Distributional solutions

Definition (Distributional solution)

Consider

$$E(\cdot)X' = A(\cdot)X + F, \tag{1}$$

with $E, A \in \mathcal{C}^\infty{}_{\mathrm{pw}}(\mathbb{R} \to \mathbb{R}^{n \times n})$, $F \in \mathbb{D}_{\mathrm{pw}W}$.
A distributional solution of (1) is

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Lemma

If $E\dot{x} = Ax + f$ has a classical solution $x : \mathbb{R} \to \mathbb{R}^n$, then $x_{\mathbb{D}}$ is a distributional solution of $EX' = AX + f_{\mathbb{D}}$.

Problems with initial value problems

Problems with IVPs

1. Writing $X(t) = x_0$ is not possible.

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has only the trivial solution (also in the distributional sense).

Solution to problem 1

For $D \in \mathbb{D}_{\mathrm{pw}W^1}$ the term $D(t-)$ is well defined.

Reason: The regular part $f_{\mathbb{D}}$ of $D = f_{\mathbb{D}} + \sum_i D_i$ is piecewise continuous.

Inconsistent initial values

Solution to problem 2

$X \in \mathbb{D}_{\text{pw}W^1}$ solves the IVP $EX' = AX + F$, $X(t_0-) = x_0$

$:\Leftrightarrow$

X solves $E_{\text{IVP}}X' = A_{\text{IVP}}X + F_{\text{IVP}}$, where

- $E_{\text{IVP}} = \mathbb{1}_{(-\infty,t_0)}0 + \mathbb{1}_{[t_0,\infty)}E$,
- $A_{\text{IVP}} = \mathbb{1}_{(-\infty,t_0)}I + \mathbb{1}_{[t_0,\infty)}A$,
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New viewpoint

An IVP is a DAE with non-smooth coefficients!

Summary

- DAEs with piecewise coefficients play an important role
  - electrical circuits with switches
  - systems with possible structural changes
  - initial value problems

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  - electrical circuits with switches
  - systems with possible structural changes
  - initial value problems
- distributional solutions must be considered
- new distributional subspaces were introduced, which
  - generalize existing approaches
  - allow for multiplication with non-smooth coefficients
  - allow for distributional IVPs
  - can deal with inconsistent initial values

Multiplication not well-defined

Counterexample

$$D = \sum_{i\in\mathbb{N}} d_n\delta_{d_n} \in \mathbb{D}\backslash\mathbb{D}_{\mathrm{pw}}, \quad d_n := \frac{(-1)^n}{n}$$

$$a = \mathbb{1}_{[0,\infty)} \in \mathcal{C}^\infty{}_{\mathrm{pw}}(\mathbb{R} \to \mathbb{R})$$

Product is not well-defined

$$aD = \sum_{k\in\mathbb{N}} \frac{1}{2k}\delta_{1/2k} \notin \mathbb{D}, \text{ because}$$

$$(aD)(\varphi) = \sum_{k\in\mathbb{N}} \frac{\varphi(1/2k)}{2k} = \pm\infty$$

for $\varphi \in \Phi$ with $\varphi(0) \neq 0$.