Partial averaging for switched DAEs with two modes

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***Abstract*— In this paper an averaging result for switched systems whose modes are represented by means of differential algebraic equations (DAEs) is presented. Homogeneous switched DAEs with periodic switchings between two modes are considered. It is proved that a (switched) averaged system can be defined also in the presence of state jumps whose amplitude does not decrease with the increasing of the switching frequency. A switched capacitor electrical circuit is considered as an illustrative example.**

I. Introduction

Averaging theory is an interesting approach for the analysis of nonlinear systems which present time scale separations [1]. A class of systems for which averaging has demonstrated its validity consists of switched systems, which can be viewed as a class of hybrid systems with a policy that at each time instant selects the active subsystem among a set of possible modes [2]. If the frequency of a periodic switching signal is high compared to the dynamics of the continuous state variables it is possible to give an explicit formulation of a (simpler) averaged model which approximates in some sense the slow dynamics of the system [3], [4]. Different aspects have been investigated in the literature dealing with the averaging for switched systems: periodic switchings [5], [6], dithering [7], hybrid behaviors [8]; for an overview see [9]. Many switched systems can be represented by means of ordinary differential equations (ODEs). On the other hand there exist practical engineering systems, such as some power electronics converters, for which this choice can be limitative and, for some topologies, the system must be represented by means of switched differential algebraic equations (DAEs) [10], [11]. Dealing with averaging for switched DAEs, in [12] it is proposed an averaged model for homogeneous switched systems which switch between two modes, while in [13] the result is extended to the case of multiple modes under the assumption of the commutativity of the consistency projectors. These results are extended in this paper by considering switched homogeneous DAEs in the presence of state jumps whose amplitude does not go to zero when the switched frequency goes to infinity. The averaging theorem presented in this paper is not a straightforward extension of the previous result because the presence of "persistent" state jumps corresponds to the relaxation of the commutativity condition of the consistency projectors. In particular, these persistent jumps can only be captured with a partial average system in the sense that it still contains switchings, but only where it is necessary to produce the remaining jumps.

The paper is organized as follows. Section II presents the class of systems under investigation and a brief reminder on the averaging for switched DAEs. The averaging in the presence of state jumps is analyzed in Sec. III. By assuming a particular structure of the projectors and of the flow matrices of DAEs, a partial averaged switched model is defined and a corresponding averaging result is proved. Illustrative examples are considered in Sec. IV: numerical results for a switched capacitor electrical circuit motivate the practical interest of the theoretical result. The conclusions are presented in Sec. V.

II. Preliminaries

Throughout the paper, let $x(t^-)$ denote the left-sided limit of x at $t \in \mathbb{R}$, i.e. $x(t^-) := \lim_{\varepsilon \searrow 0} x(t-\varepsilon)$; the right-sided limit $x(t^+)$ is defined analogously.

A. Regular Matrices

A matrix pair (E, A) with $E, A \in \mathbb{R}^{m\times n}$ is called regular, if $m = n$ and the polynomial $\det(sE - A)$ is not the zero polynomial. The following result is well known.

***Proposition** 1 (Quasi-Weierstrass [14], cf. [15], [16]):* A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is regular if and only if there exist invertible transformation matrices $S, T \in \mathbb{R}^{n\times n}$ which put (E, A) into quasi Weierstrass form

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right)$$

where $N \in \mathbb{R}^{n_2\times n_2}$, with $0 \le n_2 \le n$ is a nilpotent matrix, $J \in \mathbb{R}^{n_1\times n_1}$ with $n_1 := n - n_2$ is some matrix and I is the identity matrix of the appropriate size.
Note that, the transformation matrices S and T can easily be obtained via the so called Wong sequences, see [16].

***Definition** 1 (Flow matrix, [17]):* Consider a regular matrix pair (E, A) and its quasi Weierstrass form. The flow matrix A^{diff} of (E, A) is given by

$$A^{\text{diff}} = T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1}.$$

***Definition** 2 (Consistency Projector):* Consider a regular matrix pair (E, A) and its quasi Weierstrass form. The consistency projector Π of (E, A) is given by

$$\Pi = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}.$$

The consistency projector has the following properties

- $\Pi^k = \Pi, \quad \forall k > 0$
- $A^{\text{diff}}\Pi = A^{\text{diff}} = \Pi A^{\text{diff}}$.

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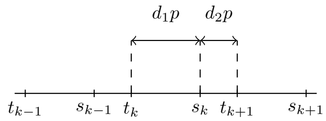

Fig. 1. Graphical representation of the time interval $[t_{k-1}, s_{k+1}]$, with $t_k = kp$, $s_k = kp + d_1 p$.

The flow matrix and the consistency projector play a fundamental role when studying the (impulse free) solutions of switched DAEs as detailed in the following.

B. Solution theory of switched DAEs

In this work, switched DAEs of the following form are considered:

$$E_1\dot{x} = A_1 x, \qquad \text{on } t \in [t_k, s_k) \tag{1a}$$
$$E_2\dot{x} = A_2 x, \qquad \text{on } t \in [s_k, t_{k+1}) \tag{1b}$$
$$x(0^-) = x_0 \in \mathbb{R}^n, \tag{1c}$$

where $E_1, A_1, E_2, A_2 \in \mathbb{R}^{n\times n}$ and, for $k \in \mathbb{N}$,

$$t_k := kp, \quad s_k := t_k + d_1 p \tag{2}$$

where $p > 0$ is the switching period and $d_1 \in (0,1)$ is the duty cycle of mode 1, while $d_2 := 1 - d_1$ is the duty cycle of mode 2, see Fig. 1. Throughout this work it is assumed that (E_1, A_1) and (E_2, A_2) are regular matrix pairs and that either the switched DAEs is impulse free or that only the impulse-free part of the (distributional) solution is of interest; for more details on impulses and distributional solutions see e.g. [18]. Note however, that impulse-freeness does not exclude jumps in the solution.

Then the (impulse-free) solution of (1) on the interval $(0, p)$ can be written as

$$x(t) = e^{A_1^{\text{diff}} t}\Pi_1 x_0, \qquad t \in (0, d_1 p),$$
$$x(t) = e^{A_2^{\text{diff}}(t-d_1 p)}\Pi_2 e^{A_1^{\text{diff}} d_1 p}\Pi_1 x_0, \qquad t \in (d_1 p, p).$$

where A_i^{diff} and Π_i, $i = 1, 2$, are the corresponding flow matrices and consistency projectors. Hence (c.f. [19]), the (impulse-free) solution behavior of the switched DAE is equivalent to the solution behavior of the following switched ODE with jumps:

$$\dot{x}(t) = A_1^{\text{diff}} x(t), \qquad t \in (t_k, s_k), \tag{3a}$$
$$x(t_k^+) = \Pi_1 x(t_k^-), \tag{3b}$$
$$\dot{x}(t) = A_2^{\text{diff}} x(t), \qquad t \in (s_k, t_{k+1}), \tag{3c}$$
$$x(s_k^+) = \Pi_2 x(s_k^-), \tag{3d}$$
$$x(0^-) = x_0 \in \mathbb{R}^n, \tag{3e}$$

with $t_0 = 0$ and the switching times given by (2).

C. Averaging for switched DAEs

***Definition** 3:* (Big O notation): Consider any functions $f : (0,\infty) \to \mathcal{V}$ and $g : (0,\infty) \to (0,\infty)$, where $\mathcal{V}$ is some normed vector space with norm $\|\cdot\|$. We say that $f(p)$ is an $\mathrm{O}(g(p))$ function ($f(p) = \mathrm{O}(g(p))$ for short), if there exist constants α and $\bar{p} > 0$ such that

$$\|f(p)\| \le \alpha g(p), \quad \forall p \in (0, \bar{p}].$$

The following averaging result for switched DAEs is already known [12]:

***Proposition** 2 (Full averaging for switched DAEs):* Consider the switched DAEs (1) with flow matrices A_1^{diff}, A_2^{diff} and consistency projectors Π_1, Π_2. Assume that the consistency projectors commute, i.e.

$$\Pi_1\Pi_2 = \Pi_2\Pi_1. \tag{4}$$

Then the corresponding averaged system is given by

$$\dot{x}_{av} = A_{av} x_{av}, \quad x_{av}(0) = \Pi_\cap x_0 \tag{5}$$

where

$$A_{av} := \Pi_\cap (A_1^{\text{diff}} d_1 + A_2^{\text{diff}} d_2)\Pi_\cap,$$
$$\Pi_\cap := \Pi_1\Pi_2,$$

and for any fixed $T > 0$ and $x_0 \in \mathbb{R}^n$ the following holds

$$\|x(t) - x_{av}(t)\| = \mathrm{O}(p), \quad \forall t \in (0, T],$$

where x denotes the (impulse-free but possibly non-continuous part of the) solution of (1) and x_{av} is the (continuously differentiable) solution of (5).

An interesting implication of the previous result is that if the projectors commute then the amplitude of the state jumps converge to zero when the switching period tends to zero, because in the limit the solution of the switched DAE coincides with a solution of an ODE (which of course does not exhibit any jumps).

***Remark** 1:* Proposition 2 makes a statement about the switched DAE (1); however, due to the equivalence of (1) with the switched ODE with jumps (3), the statement is valid also for switched ODE with jumps of the form (3). For this it is not necessary that A_i^{diff} and Π_i^{diff}, $i = 1, 2$, are defined in terms of regular matrix pairs (E_i, A_i); it suffices that the following properties hold: $\Pi_i^2 = \Pi_i$, $\Pi_i A_i^{\text{diff}} = A_i^{\text{diff}} = A_i^{\text{diff}}\Pi_i$, $i = 1, 2$, and $\Pi_1\Pi_2 = \Pi_2\Pi_1$.

III. Main result on partial averaging

If the commutativity condition (4) does not hold it is still of interest, whether a simpler averaged system can be found whose solution is an approximation of the original switched DAE. In [12] it was shown via a simple example, that without the commutativity assumption one cannot expect an averaged system which produces continuous solutions. However, there might be parts of the state variables which can be approximated with continuous solutions of a partial averaged system and for the remaining states it is not possible to “averaging away” the switchings, i.e. some switching is still needed.

***Theorem** 1:* Consider the switched system (1). Assume the following conditions hold.

(i) The matrix pairs (E_i, A_i), $i = 1, 2$, are regular with corresponding consistency projectors Π_i and flow matrices A_i^{diff} having the following structures

$$A_i^{\text{diff}} = \begin{bmatrix} A^{\text{diff}}_{11_{i[\alpha\times\alpha]}} & 0_{[\alpha\times(n-\alpha)]} \\ A^{\text{diff}}_{21_{i[(n-\alpha)\times\alpha]}} & A^{\text{diff}}_{22_i[(n-\alpha)\times(n-\alpha)]} \end{bmatrix},$$

$$\Pi_i = \begin{bmatrix} \Pi_{11_{i[\alpha\times\alpha]}} & 0_{[\alpha\times(n-\alpha)]} \\ \Pi_{21_{i[(n-\alpha)\times\alpha]}} & \Pi_{22_i[(n-\alpha)\times(n-\alpha)]} \end{bmatrix},$$

where $\alpha < n$ is independent of mode i.

(ii) The consistency projectors Π_1 and Π_2 commute partially in the sense that

$$\Pi_{\cap_p} := \Pi_{11_1}\Pi_{11_2} = \Pi_{11_2}\Pi_{11_1}.$$

In particular the matrix $\Pi_{\cap_p}$ is a projector.

(iii) The matrix $\Pi_{22_2}\Pi_{22_1}$ is a projector.

(iv) $\Pi_{22_1}\Pi_{21_2} = 0$ and $\Pi_{22_2}\Pi_{21_1} = 0$.

Let

$$A_{pav} := \Pi_{\cap_p}(A^{\text{diff}}_{11_1} d_1 + A^{\text{diff}}_{11_2} d_2)\Pi_{\cap_p}, \tag{6}$$

and consider the following

$$A^{\text{diff}}_{pav_1} := \begin{bmatrix} A_{pav} & 0 \\ A^{\text{diff}}_{21_1} & A^{\text{diff}}_{22_1} \end{bmatrix} \quad A^{\text{diff}}_{pav_2} := \begin{bmatrix} A_{pav} & 0 \\ A^{\text{diff}}_{21_2} & A^{\text{diff}}_{22_2} \end{bmatrix}$$

$$\Pi_1^* := \begin{bmatrix} I_\alpha & 0_{[\alpha\times(n-\alpha)]} \\ \Pi_{21_1} & \Pi_{22_1} \end{bmatrix} \quad \Pi_2^* := \begin{bmatrix} I_\alpha & 0_{[\alpha\times(n-\alpha)]} \\ \Pi_{21_2} & \Pi_{22_2} \end{bmatrix}$$

$$\Pi_\cap^* := \begin{bmatrix} \Pi_{11_1}\Pi_{11_2} & 0 \\ \Pi_{21_1} & \Pi_{22_1} \end{bmatrix}$$

then the corresponding switched ODE with jumps is given by

$$\dot{x}_{pav}(t) = A^{\text{diff}}_{pav_1} x_{pav}(t), \qquad t \in (t_k, s_k) \tag{7a}$$
$$x_{pav}(t_k^+) = \Pi_1^* x_{pav}(t_k^-) \tag{7b}$$
$$\dot{x}_{pav}(t) = A^{\text{diff}}_{pav_2} x_{pav}(t), \qquad t \in (s_k, t_{k+1}) \tag{7c}$$
$$x_{pav}(s_k^+) = \Pi_2^* x_{pav}(s_k^-) \tag{7d}$$
$$x_{pav}(0-) = \Pi_\cap^* x_0 \tag{7e}$$

with the switching times given by (2). Then for any given $T > 0$ and $x_0 \in \mathbb{R}^n$ the following holds

$$\|x(t) - x_{pav}(t)\| = \mathrm{O}(p), \quad \forall t \in (0, T]$$

where $x_{pav}(t)$ and $x(t)$ are the solutions of the switched partial averaged (7) and switched DAE (1) systems, respectively.

Note that the first α components of x_{pav} are governed by a non-switched ODE without jumps. In order to prove the main result the following two lemmas are needed.

Lemma 1: Consider the switched DAE (1) satisfying assumptions (i) and (ii) from Theorem 1. Let A_{pav} be given by (6) and consider the following averaged system

$$\dot{z}_{pav} = A_{pav} z_{pav}, \quad z_{pav}(0) = \Pi_{\cap_p}[I_\alpha\ 0]x_0. \tag{8}$$

Then for any $T > 0$ and $x_0 \in \mathbb{R}^n$ the following holds

$$\|[I_\alpha\ 0]x(t) - z_{pav}(t)\| = \mathrm{O}(p), \quad \forall t \in (0, T],$$

where $x(t)$ and $z_{pav}(t)$ are the solutions of (1) and (8), respectively.

Proof: Let $z := [I_\alpha\ 0]x$, then assumption (i) implies that z is governed by the following switched ODE with jumps:

$$\begin{aligned} \dot{z}(t) &= A_{11_1} z(t), & t &\in (t_k, s_k), \\ z(t_k^+) &= \Pi_{11_1} z(t_k^-), \\ \dot{z}(t) &= A_{11_2} z(t), & t &\in (s_k, t_{k+1}), \\ z(s_k^+) &= \Pi_{11_2} z(s_k^-), \\ z(0^-) &= [I_\alpha\ 0]x_0, \end{aligned}$$

with the switching times given by (2). Now invoking assumption (ii) the proof follows from Remark 1. ∎

Lemma 2: Consider a switched ODE with jumps given by

$$\dot{w}(t) = F_1 w(t) + G_1 u(t), \quad t \in (t_k, s_k) \tag{9a}$$
$$w(t_k^+) = P_1 w(t_k^-) + Q_1 v_{t_k} \tag{9b}$$
$$\dot{w}(t) = F_2 w(t) + G_2 u(t), \quad t \in (s_k, t_{k+1}) \tag{9c}$$
$$w(s_k^+) = P_2 w(s_k^-) + Q_2 v_{s_k} \tag{9d}$$
$$w(0^-) = 0, \tag{9e}$$

with the switching times given by (2) and matrices $F_1, P_1, F_2, P_2 \in \mathbb{R}^{\ell\times\ell}$, $G_1, Q_1, G_2, Q_2 \in \mathbb{R}^{\ell\times q}$. Assume that for any fixed $T > 0$ the following conditions hold

(i) $u(t) = \mathrm{O}(p)$ for all $t \in (0, T]$,

(ii) $v_{t_k} = \mathrm{O}(p)$ and $v_{s_k} = \mathrm{O}(p)$ for all $k \in [0, T/p]$,

(iii) $P_2 P_1$ is a projector,

(iv) $P_2 Q_1 = 0$ and $P_1 Q_2 = 0$.

Then for all $t \in (0, T]$ it is $w(t) = \mathrm{O}(p)$.

Proof: The solution of (9) on the interval (s_{k-1}, t_k) evaluated at t_k^- can be written as

$$w(t_k^-) = e^{F_2 d_2 p} w(s_{k-1}^+) + \int_{s_{k-1}}^{t_k} e^{F_2(t_k-\xi)} G_2 u(\xi)\mathrm{d}\xi. \tag{10}$$

Furthermore

$$w(s_{k-1}^+) = P_2 w(s_{k-1}^-) + Q_2 v_{s_{k-1}} \tag{11}$$

where $w(s_{k-1}^-)$ is the solution of (9) on the interval (t_{k-1}, s_{k-1}) evaluated at s_{k-1}^- given by

$$w(s_{k-1}^-) = e^{F_1 d_1 p} w(t_{k-1}^+) + \int_{t_{k-1}}^{s_{k-1}} e^{F_1(s_{k-1}-\xi)} G_1 u(\xi)\mathrm{d}\xi \tag{12}$$

and

$$w(t_{k-1}^+) = P_1 w(t_{k-1}^-) + Q_1 v_{t_{k-1}}. \tag{13}$$

Combining (10), (11), (12) and (13) one obtains a linear discrete time system

$$w(t_k^-) = H(p) w(t_{k-1}^-) + N(p) v_{k-1} + \mathfrak{I}(p)\{u_{k-1}\} \tag{14}$$

with

$$v_{k-1} := \begin{bmatrix} v_{t_{k-1}} \\ v_{s_{k-1}} \end{bmatrix},$$
$$u_{k-1} : [0,p] \to \mathbb{R}^q,\ t \mapsto u(t+t_{k-1}),$$
$$H(p) = e^{F_2 d_2 p} P_2 e^{F_1 d_1 p} P_1,$$
$$N(p) = \begin{bmatrix} e^{F_2 d_2 p} P_2 e^{F_1 d_1 p} Q_1 & e^{F_2 d_2 p} Q_2 \end{bmatrix},$$
$$\mathfrak{I}(p)\{u_{k-1}\} = e^{F_2 d_2 p} P_2 \int_0^{d_1 p} e^{F_1(d_1 p - \xi)} G_1 u_{k-1}(\xi) \mathrm{d}\xi + \int_{d_1 p}^{p} e^{F_2(p-\xi)} G_2 u_{k-1}(\xi) \mathrm{d}\xi.$$

The solution of (14) for all $k \in \mathbb{N}$ can be written as

$$w(t_k^-) = \sum_{i=0}^{k-1} H(p)^{k-1-i} (N(p) v_i + \mathfrak{I}(p)\{u_i\}). \tag{15}$$

Taking into account that $P_1^2 = P_1$ the equation (15) can be rewritten as

$$w(t_k^-) = \sum_{i=0}^{k-2} H(p)^{k-1-i} (P_1 N(p) v_i + P_1 \mathfrak{I}(p)\{u_i\}) + N(p) v_{k-1} + \mathfrak{I}(p)\{u_{k-1}\}.$$

Consider the Taylor approximation of the exponential matrix one obtains

$$H(p) = P_2 P_1 + \mathrm{O}(p),$$
$$N(p) = \begin{bmatrix} P_2 Q_1 + \mathrm{O}(p) & Q_2 + \mathrm{O}(p) \end{bmatrix},$$

in particular, invoking assumption (iv),

$$P_1 N(p) = [\mathrm{O}(p)\ \mathrm{O}(p)].$$

Furthermore, invoking (iii) and [13, Lemma 2],

$$H(p)^j = \mathrm{O}(1) \quad \forall j \in [0, T/p].$$

Finally, taking into account the general bound $\| \int_a^b f \| \leq (b-a)\|f\|_\infty$,

$$\mathfrak{I}(p)\{u_j\} = \mathrm{O}(p)\|u_j\|_\infty \quad \forall j \in [0, T/p].$$

Hence it follows, together with assumptions (i) and (ii),

$$w(t_k^-) = (k-1)\mathrm{O}(p^2) + \mathrm{O}(p) \quad \forall k \in [0, T/p].$$

This shows

$$w(t_k^-) = \mathrm{O}(p) \text{ as well as } w(t_k^+) = \mathrm{O}(p). \tag{16}$$

It remains to be shown that $w(t) = \mathrm{O}(p)$ for all remaining $t \in (0, T]$. Consider a given fixed time instant $t^* = t_k + \tau$ where $\tau \in [0, p)$ and $0 < t_k < T$, then the solution of (9) on the interval (t_k, τ) can be written as follows

$$w(\tau^-) = e^{F_1(\tau - t_k)} w(t_k^+) + \int_{t_k}^{\tau} e^{F_1(\tau-\xi)} G_1 u(\xi) d_\xi,\ \tau \in (0, d_1 p); \tag{17a}$$

$$w(\tau^-) = e^{F_2(\tau - s_k)} w(s_k^+) + \int_{s_k}^{\tau} e^{F_2(\tau-\xi)} G_2 u(\xi) d_\xi, \tau \in (d_1 p, p). \tag{17b}$$

Considering the following Taylor expression

$$e^{F_i(\tau - t_k)} = I + \mathrm{O}(\tau - t_k) = I + \mathrm{O}(p) \quad i = 1, 2$$

where the latter equality holds because

$$\tau - t_k \leq d_i p \quad i = 1, 2.$$

Then (17) becomes

$$w(\tau^-) \leq (I + \mathrm{O}(p)) w(t_k^+) + d_1 p \max_{\xi \in [t_k, \tau]} \|e^{F_1(\tau-\xi)} G_1 u(\xi)\| \ \tau \in [0, d_1 p);$$
$$w(\tau^-) \leq (I + \mathrm{O}(p)) w(s_k^+) + d_2 p \max_{\xi \in [s_k, \tau]} \|e^{F_2(\tau-\xi)} G_2 u(\xi)\| \ \tau \in [d_1 p, p).$$

Then by applying (i) one obtains

$$w(\tau^-) \leq (I + \mathrm{O}(p)) w(t_k^+) + \mathrm{O}(p^2), \quad \tau \in [0, d_1 p);$$
$$w(\tau^-) \leq (I + \mathrm{O}(p)) w(s_k^+) + \mathrm{O}(p^2), \quad \tau \in [d_1 p, p).$$

By (16) one obtains that $w(\tau) = \mathrm{O}(p)$ for all $\tau \in [0, p)$ and the proof is complete. ■

Proof of Theorem 1. Directly follows by combining Lemma 1 and Lemma 2. ■

Remark 2: Lemma 2 is similar to classical input-to-state-stability (ISS) results, in the sense that a small input (of order O(p)) results in a small state (also $\mathrm{O}(p)$) on any fixed time interval. Recently, an ISS result utilizing averaging for general hybrid systems has been investigated in [20].

IV. Illustrative Examples

A. Switched capacitor electrical circuit

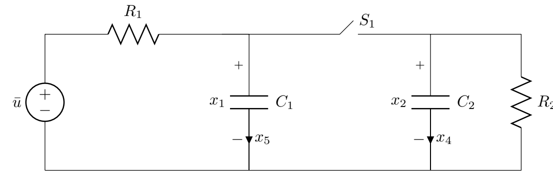

Fig. 2. A switched capacitor electrical circuit.

The proposed approach can be useful for the analysis of practical engineering systems such as the switched capacitor electrical circuit represented in Fig. 2. Consider as state variables the voltages on the capacitors and the currents through the capacitors, as shown in Fig. 2. Assume the input u is constant with amplitude $\bar{u}$, then the input can be reinterpreted as a state variable x_3 given by $\dot{x}_3 = \dot{u} = 0$ and $x_3(0) = \bar{u}$. The circuit can be represented with the following modes

Mode 1	Mode 2
$C_1\dot{x}_1 = x_5$	$C_1\dot{x}_1 = x_5$
$C_2\dot{x}_2 = x_4$	$C_2\dot{x}_2 = x_4$
$\dot{x}_3 = 0$	$\dot{x}_3 = 0$
$x_4 = -x_2/R_2$	$x_1 = x_2$
$x_1 = x_3 - R_1 x_5$	$x_1 = -R_1/R_2 x_2 + x_3 - R_1 x_4 - R_1 x_5$

with $x_0 = [0\ 0\ \bar{u}\ 0\ 0]^\top$. The two modes of the circuit depend on the position of the switch. Define the following constants

$$\rho = \frac{(R_2 + R_1)}{R_2 R_1 (C_1 + C_2)}, \quad \eta = \frac{1}{(R_1(C_1 + C_2))}.$$

Then the flow-matrices can be written as

$$A_1^{\text{diff}} = \begin{bmatrix} -1/C_1R_1 & 0 & 1/(C_1R_1) & 0 & 0 \\ 0 & -1/C_2R_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1/C_2R^2 & 0 & 0 & 0 \\ -1/C_1R_1^2 & 0 & -1/C_1R_1^2 & 0 & 0 \end{bmatrix}$$

$$A_2^{\text{diff}} = \begin{bmatrix} -\frac{C_1\rho}{(C_1+C_2)} & -\frac{C_2\rho}{(C_1+C_2)} & \eta & 0 & 0 \\ -\frac{C_1\rho}{(C_1+C_2)} & -\frac{C_2\rho}{(C_1+C_2)} & \eta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{C_1C_2\rho^2}{(C_1+C_2)} & -\frac{C_2^2\rho^2}{(C_1+C_2)} & -C_2(R_2+R_1)\eta^2 & 0 & 0 \\ \frac{C_1^2\rho^2}{(C_1+C_2)} & -\frac{C_2C_1\rho^2}{(C_1+C_2)} & -C_1(R_2+R_1)\eta^2 & 0 & 0 \end{bmatrix}$$

and the consistency projectors are given by

$$\Pi_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1/R_2 & 0 & 0 & 0 \\ -1/R_1 & 0 & 1/R_1 & 0 & 0 \end{bmatrix}$$

$$\Pi_2 = \begin{bmatrix} \frac{C_1}{(C_1+C_2)} & \frac{C_2}{(C_1+C_2)} & 0 & 0 & 0 \\ \frac{C_1}{(C_1+C_2)} & \frac{C_2}{(C_1+C_2)} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -\frac{C_1C_2\rho}{(C_1+C_2)} & -\frac{C_2^2\rho}{(C_1+C_2)} & -C_2\eta & 0 & 0 \\ -\frac{C_1^2\rho}{(C_1+C_2)} & -\frac{C_2C_1\rho}{(C_1+C_2)} & -C_1\eta & 0 & 0 \end{bmatrix}.$$

where one has

$$\Pi_{11_1}\Pi_{11_2} = \Pi_{11_2}\Pi_{11_1}, \quad \Pi_{11_1}, \Pi_{11_2} \in \mathbb{R}^3$$
$$\Pi_{22_1}\Pi_{21_2} = 0 \quad \Pi_{22_2}\Pi_{21_1} = 0.$$

Then the switched partial averaged system can be formulated as (6)-(7) were the matrix A_{pav} is given by the following

$$\begin{bmatrix} -\frac{d_1}{C_1R_1} - \frac{C_1\rho d_2}{(C_1+C_2)} & -\frac{C_2\rho d_2}{(C_1+C_2)} & \frac{d_1}{(C_1R_1)} + \eta d_2 \\ -\frac{C_1\rho}{(C_1+C_2)d_2} & -\frac{d_1}{C_2R_2} - \frac{C_2\rho d_2}{(C_1+C_2)} & \eta d_2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Consider the following circuit parameters: $C_1 = 80.36mF$, $C_2 = 8.2mF$, $R_2 = 20\Omega$, $R_1 = 10\Omega$, $\bar{u} = 5V$. In Fig. 3 the evolutions of x_1, x_2, x_4, x_5 are shown while the state variable x_3 is constant and equal to $\bar{u}$. By decreasing the switching period from $p = 0.1s$ to $p = 0.01s$ the solution of the switched DAE and that of the averaged model become close to each other.

B. Importance of Assumption (iv) in Theorem 1

Assumption (iv) in Theorem 1 is crucial for our partial averaging result. We illustrate this with the following example which does only satisfy Assumptions (i)–(iii) of Theorem 1.

Consider the following matrix pairs (E_i, A_i) with $i = 1, 2$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{3}{4} & 3 \\ 3 & -\frac{1}{4} & 1 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 0 & 5 & 0 \\ 0 & 7 & 0 \\ -3 & 1 & 0 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 5 & -2 & 0 \\ 6 & -1 & 0 \\ 1 & 3 & 0 \end{bmatrix}.$$

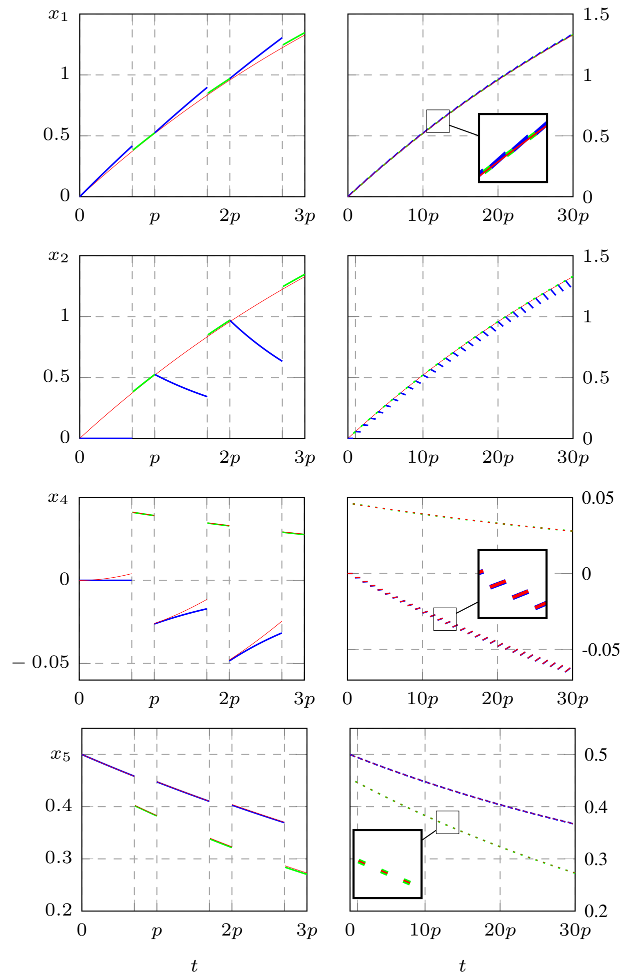

Fig. 3. Time evolutions of state components for slow switching ($p = 0.1s$, left) and fast switching ($p = 0.01s$, right). The partial averaging evolutions are plotted in red while the trajectories of the switched DAE are colored according to the active mode (mode 1 blue, mode 2 green).

The flow-matrices are the following

$$A_1^{\text{diff}} = \begin{bmatrix} -\frac{45}{49} & 0 & 0 \\ \frac{405}{2401} & 0 & 0 \\ -\frac{3711}{9604} & 0 & 0 \end{bmatrix} \qquad A_2^{\text{diff}} = \begin{bmatrix} \frac{17}{3} & 0 & 0 \\ -\frac{17}{9} & 0 & 0 \\ \frac{74}{9} & 0 & 0 \end{bmatrix}$$

while the consistency projectors are given by

$$\Pi_1 = \begin{bmatrix} 1 & 0 & 0 \\ -0.18 & 0 & 0 \\ -0.05 & -0.25 & 1 \end{bmatrix} \qquad \Pi_2 = \begin{bmatrix} 1 & 0 & 0 \\ -0.33 & 0 & 0 \\ 0.33 & 1 & 1 \end{bmatrix}$$

where one has:

$$\Pi_{11_1}\Pi_{11_2} = \Pi_{11_2}\Pi_{11_1}, \quad \Pi_{11_1}, \Pi_{11_2} \in \mathbb{R} \tag{21}$$

$$\Pi_{22_1}\Pi_{21_2} = \begin{bmatrix} 0 \\ 0.42 \end{bmatrix}, \quad \Pi_{22_2}\Pi_{21_1} = \begin{bmatrix} 0 \\ -0.23 \end{bmatrix}. \tag{22}$$

Fig. 4 shows the evolutions of the first two state variables. By decreasing the switching period the solution of the switched DAE and that of the averaged model become close

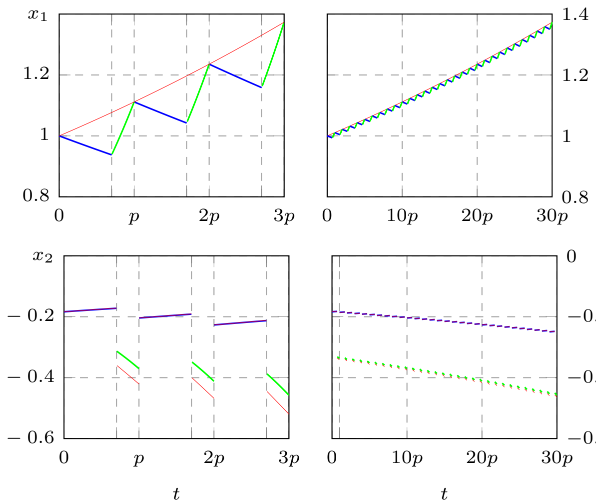

Fig. 4. Evolutions of the first two state components for slow switching ($p = 0.1s$, left) and fast switching ($p = 0.01s$, right). The averaging dynamics are plotted with red line, while the trajectories of the switched DAE are colored according to the active mode (mode 1 blue, mode 2 green).

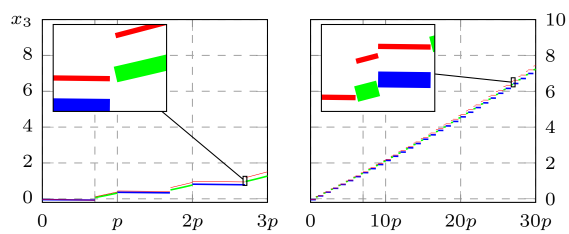

Fig. 5. Time evolutions of x_3 for slow switching ($p = 0.1s$, left) and fast switching ($p = 0.01s$, right). The partial averaging evolution is plotted in red while the trajectory of the switched DAE is colored according to the active mode (mode 1 blue, mode 2 green).

to each other. Due to (22) the averaging result doesn't hold for the third state variable, see Fig. 5. It can be seen that the averaged dynamic itself do not converge (the growth rate increases with increasing the switching frequency). The set of consistency projectors is not product unbounded as regard the third variable, then the state x_3 grows unbounded on a fixed time-interval with a switching frequency going to infinity, [19]. Furthermore the error between the solution of the switched DAE and that of the averaged model doesn't decrease by increasing the switching frequency. In Fig. 5 is highlighted that the absolute error between the averaged and the switched dynamics remain the same for both switching periods while the relative error with respect to the value of the state variable decreases by choosing lower values of p.

V. Conclusions

The averaging problem for switched systems with periodic commutations between two modes, each one described by homogeneous linear differential algebraic equations (DAEs) has been considered. It is shown that also in the presence of state jumps an averaged approach is possible and the solution of the switched system converges to the partial averaged switched system with an error of the order of the switching period. A switched capacitor electrical circuit motivates the practical interest of the proposed analysis. A possible direction for future research is the extension of the averaging result to the case of switched DAEs with multiple modes.

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