Indiscernible topological variations in DAE networks with applications to power grids [★]

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Abstract: The ability to detect topology variations in dynamical networks defined by differential algebraic equations (DAEs) is considered. We characterize the existence of initial states, for which topological changes are indiscernible. A key feature of our characterization is the ability to verify indiscernibility just in terms of the nominal topology. We apply the results to a power grid model and also discuss the relationship to recent mode-detection results for switched DAEs.

Keywords: dynamical networks, time-varying topologies, mode-detection, switched systems, descriptor systems

1. INTRODUCTION

Control theory of networks and multiagent systems has gained enormous popularity in the last years, because it has numerous important applications (Olfati-Saber and Murray, 2004; Beard et al., 2006; Scardovi and Sepulchre, 2009), and many unsolved mathematical questions. The dynamics of such networks are governed by the underlying topology (given by the edges of a graph) with predefined coupling rules and the dynamics of each agent at the nodes of the graph.

One important question in this context is, whether a topological change in the form of a removal or addition of an edge always has an effect on the dynamics. If not, this may lead to severe problems in applications because faults or attacks may stay unnoticed until much later, leading to performance losses or even to instability phenomena.

The problem of detecting topology variations in dynamical networks has therefore gained much attention in the last years (Rahimian et al., 2012; Rahimian and Preciado, 2014; Rahimian et al., 2014; Torres et al., 2015; Battistelli and Tesi, 2015, 2016). In all the aforementioned works, however, the analysis is confined to networks whose dynamics can be described via ordinary differential equations (ODEs). On the other hand, there are no results dealing with networks of differential-algebraic equations (DAEs). Networks of DAEs arise in several applications of practical interest, examples being electrical and water distribution networks, where the algebraic equations describe laws of conservation of mass, energy and current.

In this note, inspired by the results in Battistelli and Tesi (2015, 2016), we consider networks of DAEs with diffusive coupling, and study under what conditions topological changes (a removal or addition of an edge) cannot be inferred from observations of the network dynamics, referring to this phenomenon as *"indiscernibility"*. We provide necessary and sufficient conditions for indiscernibility that can be checked by only looking at the eigenvalues/eigenvectors of the nominal network configuration. This property is extremely appealing since it avoids the need to determine eigenvalues/eigenvectors of the various "faulty" topologies. Interestingly, the proposed analysis is general enough to include the case where each network node obeys different dynamics (and has possibly different state dimension).

The results presented here consider discernibility based on the whole state trajectory. This is just a first step, because once a topological change results in a change of the dynamics, the next question is, whether this change can actually be seen by a limited amount of sensors in the network. This problem has been widely studied in the general framework of switched systems; however, these results do not take into account the special structure of topological changes and it is a topic of future research to consider discernibility also for networks with a limited amount of measurements.

One should mention that the problem of detecting topological changes can also be cast as a topology identification problem; for example, see Materassi and Innocenti (2010); Sanandaji et al. (2011); Chowdhary et al. (2011). In fact, detection and identification are certainly closely connected problems. However, identification does not assume prior knowledge of the nominal network configuration. This information is crucial in order to relate nominal and modified network configurations, and to provide conditions on discernibility that can be checked by only looking at the properties of the nominal configuration.

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[1] This work was carried out while the second author was at the University of Kaiserslautern.

This note is structured as follows: We first introduce a nominal network of DAEs and the resulting model of the overall dynamics (which by itself is a large DAE) together with the possible topological changes. Afterwards we formally define discernibility and state two main results highlighting an important connection between discernibility and certain eigenvalue-eigenvector pairs (Theorem 2) which leads to a very simple characterization for discernibility (Theorem 5) that can be checked solely in terms of the nominal network configuration. Afterwards, we consider the special case where all agents have the same dynamics, and we show that discernibility only depends on the graph topology (Theorem 8). In Section 5 we apply our results to a power grid model. Finally, we relate our results with mode-detection notions for switched DAEs in Section 6.

Due to space limitations, the proofs are omitted.

2. PROBLEM STATEMENT

For a finite index set $\mathfrak{V} = \{1, 2, \ldots, N\}$, $N \in \mathbb{N}$, we consider a family of differential algebraic equations (DAEs), $i \in \mathfrak{V}$,

$$E_i \dot{x}_i = A_i x_i + b_i u_i,$$
$$y_i = c_i x_i, \tag{1}$$

where $E_i, A_i \in \mathbb{R}^{n_i \times n_i}$, $n_i \in \mathbb{N}$, $b_i, c_i^\top \in \mathbb{R}^{n_i}$. These DAEs are connected via a network given by an undirected graph $\mathfrak{G} = (\mathfrak{V}, \mathfrak{E})$ with $\mathfrak{E} \subseteq \mathfrak{V} \times \mathfrak{V}$ and via diffusive coupling of the form

$$u_i = \sum_{k:(i,k)\in\mathfrak{E}} w_{ik}(y_k - y_i), \tag{2}$$

where $w_{ij} > 0$ with $w_{ji} = w_{ij}$ for $i, j \in \mathfrak{V}$. Let $x = \mathrm{col}\{x_1, \ldots, x_N\}$ represent the overall state of the network. The collective dynamics can be written in compact form as

$$\mathcal{E}\dot{x} = (\mathcal{A} - \mathcal{B}\mathfrak{L}\mathcal{C})x =: \mathcal{A}_\mathfrak{L} x \tag{3}$$

where $\mathcal{E} = \mathrm{diag}\{E_1, \ldots, E_N\}$, $\mathcal{A} = \mathrm{diag}\{A_1, \ldots, A_N\}$, $\mathcal{B} = \mathrm{diag}\{b_1, \ldots, b_N\}$, $\mathcal{C} = \mathrm{diag}\{c_1, \ldots, c_N\}$, and where $\mathfrak{L} = [\ell_{i,j}]_{i,j\in\mathcal{V}}$ with

$$\ell_{i,j} = \begin{cases} -w_{i,j}, & i \neq j, (i,j) \in \mathfrak{E}, \\ 0, & i \neq j, (i,j) \notin \mathfrak{E}, \\ \displaystyle\sum_{k:(i,k)\in\mathfrak{E}} w_{i,k}, & i = j. \end{cases} \tag{4}$$

Note that $\mathfrak{L}$ is the weighted *Laplacian* of $\mathfrak{G}$, in particular, $\mathfrak{L}$ is symmetric and positive semidefinite. The simplest case for (3) is the case $n_i = 1$, $E_i = 1$, $A_i = 0$, $b_i = 1$, $c_i = 1$ resulting in the overall system

$$\dot{x} = -\mathfrak{L}x.$$

This case was studied in (Battistelli and Tesi, 2015) and our goal is to expand the results therein to the more general case above. An interesting example for the general structure (3) is a model of a power grid, see Section 5 for details.

Our goal is to study the effect of topological variations, i.e., the removal (or addition) of an interconnection of the network. In terms of the (weighted) Laplacian the removal of an edge $(i,j) \in \mathcal{E}$ can be written as a rank-one change as follows:

$$\overline{\mathfrak{L}} = \mathfrak{L} - w_{ij}(e_i - e_j)(e_i - e_j)^\top, \tag{5}$$

where $e_i \in \mathbb{R}^n$ is the i-th unit vector. Connecting two previously unconnected nodes leads to a similar rank-one change:

$$\overline{\mathfrak{L}} = \mathfrak{L} + w_{ij}(e_i - e_j)(e_i - e_j)^\top.$$

We are now interested in determining the existence of nontrivial initial values such that the solution x of (3) is identical to a solution $\overline{x}$ of

$$\mathcal{E}\dot{\overline{x}} = (\mathcal{A} - \mathcal{B}\overline{\mathfrak{L}}\mathcal{C})\overline{x} =: \mathcal{A}_{\overline{\mathfrak{L}}}\overline{x}. \tag{6}$$

If this is the case, we are not able to detect the topological change (*e.g.*, induced by a fault or by an attack) *even if we have full knowledge of the state of the system*. Only if we can give a negative answer to the above question, it makes sense to further study the mode detection problem where we only have a subset of the state available for measurement (which is not considered in this note and is a topic of future research).

3. INDISCERNIBLE TOPOLOGICAL CHANGES

Definition 1. (Indiscernible topological changes). A topological change of (3) resulting in (6) is called *possibly-indiscernible* iff it has at least one indiscernible initial state $x_0 \in \mathbb{R}^n \setminus \{0\}$, i.e., there exists a solution x of (3) with $x(0) = x_0$ which is also a solution of (6). Otherwise, the topological change is called discernible. □

Note that one could also introduce the stronger notion of indiscernible topological changes (and its weaker opposite: possibly-discernible), which means that the topological change cannot be detected no matter what the initial state is (all initial states are indiscernible); however this is not such a useful definition in the context considered here.

We first relate indiscernible initial states with common eigenvalue-eigenvector pairs of $(\mathcal{E}, \mathcal{A}_\mathfrak{L})$ and $(\mathcal{E}, \mathcal{A}_{\overline{\mathfrak{L}}})$.

Theorem 2. Consider (3) and (6) and assume that $(\mathcal{E}, \mathcal{A}_\mathfrak{L})$ and $(\mathcal{E}, \mathcal{A}_{\overline{\mathfrak{L}}})$ are regular matrix pairs, i.e., $\det(s\mathcal{E} - \mathcal{A}_\mathfrak{L})$ and $\det(s\mathcal{E} - \mathcal{A}_{\overline{\mathfrak{L}}})$ are nonzero polynomials in s. Then there exists a nontrivial indiscernible initial state if, and only if, there exists a common eigenvalue-eigenvector pair $(\lambda, v) \in \mathbb{C} \times \mathbb{C}^n \setminus \{0\}$ of $(\mathcal{E}, \mathcal{A}_\mathfrak{L})$ and $(\mathcal{E}, \mathcal{A}_{\overline{\mathfrak{L}}})$, i.e.,

$$(\lambda\mathcal{E} - \mathcal{A}_\mathfrak{L})v = 0 = (\lambda\mathcal{E} - \mathcal{A}_{\overline{\mathfrak{L}}})v.$$

In particular, any common eigenvalue-eigenvector pair (λ, v) leads to an indiscernible initial state given by the real part of v.

Remark 3. (Regularity of (3)). The restriction to regular DAEs (3) is needed to have existence and uniqueness of solutions and is also needed to have a well-defined notion of eigenvalues and eigenvectors. Note however, that we do *not* assume that each agent has dynamics defined by a regular DAE; in fact our two forthcoming examples show that it may be a common situation to have non-regular DAEs for the node dynamics. On the other hand, coupling regular DAEs does not necessarily lead to regularity of the overall DAE system. Altogether, the problem of characterizing regularity of (3) is extremely important; however, we are not aware of simple conditions which guarantee regularity of the coupled system. □

We now highlight a simple characterization for the existence of a common eigenvalue-eigenvector pair.
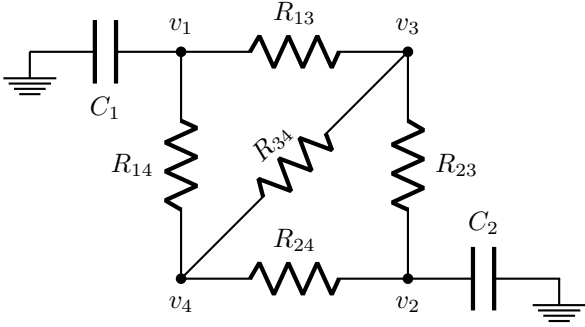
Fig. 1. Simple RC-circuit.

Lemma 4. Let $(\lambda, v) \in \mathbb{C} \times \mathbb{C}^n \setminus \{0\}$ be an eigenvalue-eigenvector pair of $(\mathcal{E}, \mathcal{A}_{\mathfrak{L}})$. Then (λ, v) is also an eigenvalue-eigenvector of $(\mathcal{E}, \mathcal{A}_{\overline{\mathfrak{L}}})$ if, and only if,

$$v \in \ker(\mathcal{A}_{\mathfrak{L}} - \mathcal{A}_{\overline{\mathfrak{L}}})$$

This simple observation now leads to our main result.

Theorem 5. Consider a family of DAEs of the form (1) connected by a network graph $\mathfrak{G} = (\mathfrak{V}, \mathfrak{E})$ with weighted Laplacian $\mathfrak{L}$ resulting in the overall system (3), which we assume to be regular. Any regularity-preserving removal/addition of the edge (i, j) is a possibly-indiscernible topological change if, and only if, either $b_i = 0$ and $b_j = 0$ or there exists an eigenvector $v \in \mathbb{C}^n \setminus \{0\}$ of $(\mathcal{E}, \mathcal{A}_{\mathfrak{L}})$ with

$$(\mathcal{C}v)_i = (\mathcal{C}v)_j.$$

Note that indiscernibility can be solely checked in terms of the nominal system parameters $(\mathcal{E}, \mathcal{A}_{\mathfrak{L}})$, in particular, it is not necessary to calculate eigenvalues/eigenvectors of the various "faulty" topologies. Another interesting implication of Theorem 5 is that there is a vast literature devoted to the computation of the eigenspace of Laplacian-dependent matrices (for example, the eigenspace of a Laplacian matrix is known analytically for many fundamental graphs). These results can be used to effectively assess the conditions dictated by Theorem 5 (cf. Section 4).

We now illustrate the result by a variant of the well known Wheatstone bridge.

Example 6. Consider an electrical RC circuit as shown in Figure 1. Here vertices 1 and 2 are dynamic vertices whereas vertices 3 and 4 lead to algebraic equations. In the framework of (1) we have

$$\begin{aligned}
\text{Node 1}: & \quad -C_1 \dot{v}_1 = u_1, & y_1 = v_1, \\
\text{Node 2}: & \quad -C_2 \dot{v}_2 = u_2, & y_2 = v_2, \\
\text{Node 3}: & \quad 0 = u_3, & y_3 = v_3, \\
\text{Node 4}: & \quad 0 = u_4, & y_4 = v_4,
\end{aligned}$$

together with the coupling conditions

$$\begin{aligned}
u_1 &= R_{14}(v_1 - v_4) + R_{13}(v_1 - v_3), \\
u_2 &= R_{24}(v_2 - v_4) + R_{23}(v_2 - v_3), \\
u_3 &= R_{13}(v_3 - v_1) + R_{23}(v_3 - v_2) + R_{34}(v_3 - v_4), \\
u_4 &= R_{14}(v_4 - v_1) + R_{24}(v_4 - v_2) + R_{34}(v_4 - v_3).
\end{aligned}$$

Altogether, we get a system given by (3) with

$$\mathcal{E} = \begin{bmatrix} -C_1 & 0 & 0 & 0 \\ 0 & -C_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \mathcal{A} = 0_{4\times4}, \ \mathcal{B} = I_4, \ \mathcal{C} = I_4$$

and

$$\mathfrak{L} = \begin{bmatrix} R_{13}+R_{14} & 0 & -R_{13} & -R_{14} \\ 0 & R_{23}+R_{24} & -R_{23} & -R_{24} \\ -R_{13} & -R_{23} & R_{13}+R_{23}+R_{34} & -R_{34} \\ -R_{14} & -R_{24} & -R_{34} & R_{14}+R_{24}+R_{34} \end{bmatrix}.$$

Then, (3) reduces to

$$\mathcal{E}\dot{v}(t) = \mathfrak{L}v(t). \tag{7}$$

Note that although the DAEs of node 3 and 4 are *not regular* the coupled system (7) is regular for all positive capacitor and resistor values. Assume now that all circuit parameters are equally one, then the matrix pair $(\mathcal{E}, \mathfrak{L})$ has the two finite eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -2$ with corresponding eigenvectors

$$v^1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad v^2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

Since there exists an eigenvector and a pair (i, j) such that the i-th and j-th entries are equal, the electrical circuits has initial values where a removal of an edge could not be detected. In fact, since all eigenvectors corresponding to the finite eigenvalues have matching 3rd and 4th entries, the removal of edge $(3, 4)$ is undetectable for *any* consistent initial value.

4. HOMOGENEOUS NETWORKS

In the homogeneous case, where all coupled systems are identical, i.e., $E_i = E$, $A_i = A$, $b_i = b$, $c_i = c$ and $n_i = n$ for all $i \in \mathfrak{V}$, the overall dynamics can be written in compact form as

$$\mathcal{E}\dot{x} = \mathcal{A}_{\mathfrak{L}}x, \tag{8}$$

where

$$\begin{aligned}
\mathcal{E} &:= (I_N \otimes E), \\
\mathcal{A}_{\mathfrak{L}} &:= (I_N \otimes A) - \mathfrak{L} \otimes bc.
\end{aligned}$$

We will show that in this special case indiscernibility is in fact independent of the system parameters and only depends on the Laplacian of the connection graph, provided a certain observability condition holds.

We first highlight the properties of eigenvalue-eigenvectors pairs of $(\mathcal{E}, \mathcal{A}_{\mathfrak{L}})$ in the homogeneous case.

Lemma 7. Let $\alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbb{R}$ be the N real eigenvalues (counting multiples) of the symmetric Laplacian $\mathfrak{L}$. Then

$$\text{spec}(\mathcal{E}, \mathcal{A}_{\mathfrak{L}}) = \bigcup_{i=1}^{N} \text{spec}(E, A - \alpha_i bc). \tag{9}$$

Furthermore, let v be a (generalized) eigenvector of $(\mathcal{E}, \mathcal{A}_{\mathfrak{L}})$ for an eigenvalue λ of $(\mathcal{E}, \mathcal{A}_{\mathfrak{L}})$ with $\lambda \in \text{spec}(E, A - \alpha bc)$ for some eigenvalue α of $\mathfrak{L}$. Then

$$v = z \otimes w \tag{10}$$

where z is an eigenvector of $\mathfrak{L}$ corresponding to α and w is a (generalized) eigenvector of $(E, A - \alpha bc)$ corresponding to λ. Moreover, if v and w are generalized eigenvectors, then they have the same rank.

Using these results, we can obtain a more specific conditions on the existence of non-null indiscernible states. Specifically, we have the following result.

Theorem 8. Consider a family of identical DAEs (1) of the form
$$E\dot{x} = Ax + bu$$
$$y = cx$$
connected via the diffusive coupling (2) by a network with weighted Laplacian $\mathfrak{L}$ resulting in the overall system (8), which we assume to be regular. Suppose furthermore that $b \neq 0$ and that (E, A, c) is observable in the behavioral sense, *i.e*, rank $\begin{bmatrix} \lambda E - A \\ c \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$, see e.g. Berger et al. (2017). Then, any regularity-preserving removal/addition of the edge (i, j) is a possibly-indiscernible topological change if, and only if, there exists an eigenvector $z \in \mathbb{C}^n \setminus \{0\}$ of $\mathfrak{L}$ such that
$$z_i = z_j.$$
In that case any such eigenvector leads to an indiscernible initial state $v = z \otimes w$ as in (10).

The Laplacian matrix $\mathfrak{L}$ always has eigenvalue zero with corresponding eigenvector $z = (1, 1, \ldots, 1)^\top$. Consequently, any topological change of a homogeneous network is possibly-indiscernible. This special eigenvector corresponds to the situation where all subsystems start with the same initial value; as a consequence, the diffusive coupling is zero and a topological variation has no effect on the dynamics. However, Theorem 8 also shows which initial values may lead to indiscernibility and it is also reasonable to expect that eigenvectors with $z_i \approx z_j$ give rise to initial values where it is theoretically possible but difficult in practice to detect topological changes.

Furthermore, we would like to stress that $b \neq 0$ is an obvious assumption (otherwise there is no coupling) and that the observability assumption on (E, A, c) is quite natural, as otherwise there would be dynamics which would not effect the output of the individual systems and therefore are also not effected by the coupling structure.

5. TOPOLOGICAL CHANGES IN POWER GRIDS

We will show now how power grids can be modelled such that they fit into our framework. In particular, we are able to determine whether there are critical removal/additions of power lines which may go undetected for certain initial states even if we would have knowledge of the whole state of the network. This may have important application in the context of the security/vulnerability of cyber-physical systems, for example, we are able to answer the question whether an attacker can change the network topology without the grid operators being able to detect that change. Another application is determining the right moment when to disconnect a power line (*e.g.*, for maintenance) such that the effect on the remaining network is negligible.

We use a power grid model described in (Groß et al., 2016), which consists of p generators and q load buses. The i-th generator is modeled as
$$\dot{\alpha}^i(t) = \omega^i(t),$$
$$M^i\dot{\omega}_i(t) = -D^i\omega^i(t) - K^i\alpha^i(t) + P_g^i(t) - P_e^i(t),$$
where $\alpha^i = (\alpha_1^i, \alpha_2^i, \ldots, \alpha_{\eta_i}^i)$, $\omega^i = (\omega_1^i, \omega_2^i, \ldots, \omega_{\eta_i}^i)$ are the (incremental) angles and angular velocities of the $\eta_i \geq 0$ rotating masses of the generator, $P_g^i =$

$(p_{g,1}^i, p_{g,2}^i, \ldots, p_{g,\eta_i}^i)$ is the generator power acting on each of the η_i rotating masses (the turbines) and $P_e^i = (0, \ldots, 0, p_e^i)$ is the electrical power acting on the last rotating mass (the actual generator); $M_i, D^i, K_i \in \mathbb{R}^{\eta_i \times \eta_i}$ are the mass, damping and stiffness matrices, respectively. Let $\theta^i(t) \in \mathbb{R}$ be the voltage angle at the generator bus, then the electrical power acting on the generator, can be approximately expressed as:
$$p_e^i(t) = \frac{1}{z_i}\left(\alpha_{\eta_i}^i(t) - \theta^i(t)\right),$$
where $z_i > 0$ is the transient reactance of the generator. Finally, the linearized power flow equations are
$$p^i(t) + p_e^i(t) = \sum_{j=1}^{p+q} b_{ij}(\theta^i(t) - \theta^j(t))$$
at the generator buses $i = 1, \ldots, p$ and
$$p^i(t) = \sum_{j=1}^{p+q} b_{ij}(\theta^i(t) - \theta^j(t))$$
at the load buses $i = p + 1, \ldots, p + q$, where $b_{ij} = b_{ji} \geq 0$ is the susceptance between bus i and j and $p^i(t)$ is the external load at each bus. Assuming constant generator power and constant loads, the generator and bus models can be brought into the structure (1) via
$$x_i = (p^i, P_g^{i\top}, \alpha^{i\top}, \omega^{i\top}, \theta^i)^\top,$$
$$E_i = \begin{bmatrix} 1 & & & & \\ & I_{\eta_i} & & & \\ & & I_{\eta_i} & & \\ & & & M^i & \\ & & & & 0 \end{bmatrix},$$
$$A_i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & I & -(K^i + \frac{1}{z_i}e_{\eta_i}e_{\eta_i}^\top) & -D^i & \frac{1}{z_i}e_{\eta_i} \\ -1 & 0 & -\frac{1}{z_i}e_{\eta_i}^\top & 0 & \frac{1}{z_i} \end{bmatrix},$$
$$B_i = e_{\eta_i}, \quad C_i = e_{\eta_i}^\top$$
for the generator buses $i = 1, \ldots, p$ and $x_i = (p^i, \theta^i)^\top$,
$$E_i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_i = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_i = [0 \; 1],$$
for the load buses $i = p + 1, \ldots, p + q$; the coupling equations for all buses are
$$u_i = \sum_{j=1}^{p+q} b_{ij}(\theta^i - \theta^j),$$
where $b_{ij} = b_{ji} \geq 0$ is the susceptance between buses i and j. In (Groß et al., 2016) it was shown that the overall DAE is regular if, and only if, each connected component of the network graph contains at least one generator bus, hence we can apply Theorem 5 to check whether there are possibly-indiscernible topological changes. Note that similar to the Wheatstone bridge example the DAEs corresponding the load buses are *not regular*.

Example 9. Consider a power grid system as in Section 5 with one generator which is connected to bus 1. Define the vector $v = (p^1, P_g^{1\top}, \alpha^{1\top}, \omega^{1\top}, \theta^1, p^2, \theta^2, \ldots, p^{q+1}, \theta^{q+1})^\top$ as follows:
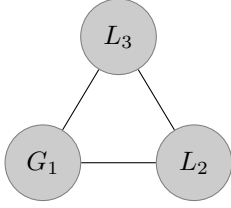
Fig. 2. Power grid of example 10

$$p^1 = \frac{1}{z_1}, \qquad P_g^1 = -\frac{1}{z_1}e_{\eta_1}, \quad \alpha^1 = 0, \quad \omega^1 = 0$$
$$p^i = 0, \; \forall i \neq 1, \quad \theta^i = 1, \; \forall i$$

Then v is an eigenvector to the eigenvalue zero of the corresponding network DAE. It satisfies

$$Cv = (1 \; 1 \; \cdots \; 1)^\top,$$

hence the system is possibly indiscernible by Theorem 5. This means it can happen in a power grid that there is an equilibrium with no flow on a particular line and thus switching of this line cannot be detected. However, as long as there is a power flow on a line, a switch of this line will affect the solution and thus be detected. □

Example 9 is artificial in the sense that the power produced at the generator is consumed at the same node, hence no power is distributed trough the network. The following example shows that indiscernible topological changes can happen even when there is a power flow in the network.

Example 10. Consider the simple power network consisting of one generator and two loads, connected as shown in Figure 2.

Then it is easily seen that there is an eigenvector v corresponding to the eigenvalue zero such the $(Cv)_2 = (Cv)_3$, i.e., there are initial values for which a disconnection of buses 2 and 3 remains undetected. These initial values correspond to the situation where each load node gets exactly the amount of power from the generator that is consumed there, i.e., when there is no power flow between buses 2 and 3. □

Example 10 points towards an obvious principle in these power grid models: A removal of a power line is undetectable if there is no power flow on this line. However, the latter situation is not an obvious necessary or sufficient condition in terms of the formalism in Theorem 5. Clarifying the connection between these conditions is a topic of future research.

6. CONNECTIONS WITH SWITCHING SIGNAL OBSERVABILITY

The problem of detecting topological changes in dynamical networks can be seen as a special case of the mode-detection problem in switched systems. It was recently highlighted in (Küsters and Trenn, 2017) that the ability to deduce the active mode from the measured output of the system (σ-observability) in all situations reduces to the ability to distinguish the outputs from systems with constant switching signals. In particular, the effect of the actual switching event (here the topological change) is not taken into account. The weaker notion of switching

time observability (t_S-observability) is concerned with the problem of detecting the time of a switch and is actually exactly the problem we studied here (see the forthcoming Lemma 12). If a switch occurs it is also of interest from which mode to which mode the system switched (*e.g.*, which power line was disconnected if a topological change is detected). This mode-detection problem for non-constant switching signals (σ_1-observability) is in general indeed weaker than σ-observability and stronger than t_S-observability.

In this section we want to investigate these three observability notions for the specially structured case considered here. It turns out (see the forthcoming Theorem 13) that all notion are in fact equivalent if the DAE-models have index one (i.e., switches do not produce Dirac impulses and the input is not differentiated).

Towards this end we first introduce the formal definitions of σ-, σ_1-, and t_S-observability of a switched DAE of the form

$$\mathcal{E}_\sigma \dot{x} = \mathcal{A}_\sigma x \qquad (11)$$

where $\sigma : \mathbb{R} \to \{0, 1, \dots, K\}$, $K \in \mathbb{N}$, is a piecewise-constant right-continuous switching signal, which is constant on $(-\infty, t_0)$ for some $t_0 > 0$. Note that a switched DAE may exhibit jumps and Dirac impulses in its solution, hence we have to consider the space of piecewise-smooth distributions (Trenn, 2009a, 2012) as the underlying solution space; in particular, x in (11) can only be evaluated at $t \in \mathbb{R}$ as left-/right-limit denoted by $x(t^-)/x(t^+)$ or as the impulsive part, denoted by $x[t]$, see Trenn (2009b) for details. If each matrix pair $(\mathcal{E}_k, \mathcal{A}_k)$, $k \in \{0, 1, \dots, K\}$ is regular, then the switched DAE (11) is uniquely solvable for any (possibly inconsistent) initial condition $x(0^-) = x_0 \in \mathbb{R}^n$ and we will denote this solution by $x_{(x_0, \sigma)}$. Note that we assume the initial condition x_0 to be consistent with the initial mode $\sigma(0)$.

When studying discernibility or mode-detection for (11) it is necessary to exclude the zero initial state, because for an identically zero state the trivial dynamics are unaffected by the current value of the switching signal. While for switched ODEs (where $\mathcal{E}_k = I$ for all k) it is sufficient to exclude $x_0 = 0$, the situation is a bit more complicated in the switched DAE case, as it is possible that the state jumps to zero later on. Therefore, we restrict our attention to the following intervals

$$s_{(x_0, \sigma)} := \left\{ t \mid x_{(x_0, \sigma)}(t^+) \neq 0 \lor x_{(x_0, \sigma)}[t] \neq 0 \right\},$$

If the state jumps to zero at time t_S, we have $s_{(x_0, \sigma)} = (-\infty, t_S)$ if there is no impulse at the switch and else $s_{(x_0, \sigma)} = (-\infty, t_S]$. The *relevant switching times* are then denoted by

$$T_{(x_0, \sigma)} := \left\{ t \mid t \text{ is a discontinuity of } \sigma \right\} \cap \overline{s_{(x_0, \sigma)}}.$$

We are now ready to formally define σ-, σ_1-, and t_S-observability.

Definition 11. The regular switched DAE (11) is called

- σ-observable :⇔ for all σ, $\tilde{\sigma}$ and all $x_0 \in \mathbb{R}^n$ it holds

$$\sigma|_{\mathcal{J}} \neq \tilde{\sigma}|_{\mathcal{J}} \; \Rightarrow \; x_{(x_0, \sigma)} \neq x_{(x_0, \tilde{\sigma})}, \qquad (12)$$

  where $\mathcal{J} := s_{(x_0, \sigma)} \cup s_{(x_0, \tilde{\sigma})}$;
- σ_1-observable :⇔ (12) holds for all $x_0 \in \mathbb{R}^n$ and for all σ, $\tilde{\sigma}$ with

$$1 \leq \min \left\{ |T_{(x_0, \sigma)}|, |T_{(x_0, \tilde{\sigma})}| \right\};$$

- t_S-observable :$\Leftrightarrow$ for all σ, $\tilde{\sigma}$ and all $x_0 \in \mathbb{R}$ it holds
$$T_{(x_0,\sigma)} \neq T_{(x_0,\tilde{\sigma})} \;\Rightarrow\; x_{(x_0,\sigma)} \neq x_{(x_0,\tilde{\sigma})}.$$
$\square$

From these definitions, we can directly conclude
$$\sigma\text{-obs.} \quad\Rightarrow\quad \sigma_1\text{-obs.} \quad\Rightarrow\quad t_S\text{-obs.}; \qquad (13)$$
the converse is not true in general, cf. Küsters and Trenn (2017) for the switched ODE case and Küsters et al. (2017) for the switched DAE case.

We now relate t_S-observability of the switched DAE (11) with possible-discernibility of each mode pair, where possible-discernibility of two modes $(\mathcal{A}_i, \mathcal{E}_i)$ and $(\mathcal{A}_j, \mathcal{E}_j)$ is defined analogously as in Definition 1.

Lemma 12. Consider the switched DAE (11). It is t_S-observable if, and only if, the modes $k \in \{0, \ldots, K\}$ are pairwise possibly-discernible.

As the conditions for σ- and σ_1-observability in Küsters et al. (2017) are rather technical, we restrict the attention here to DAEs having index one, i.e. the nilpotent matrix in the quasi-Weierstrass form (see e.g. Berger et al. (2012)) of $(\mathcal{E}, \mathcal{A})$ is zero or, equivalently, rank $\mathcal{E} = \deg \det(s\mathcal{E} - \mathcal{A})$. Note that the power grid model considered here has this property.

Theorem 13. Consider the switched DAE (11) and assume that each mode of the system (11) has index one. Then t_S-, σ_1- and σ-observability are all equivalent and coincide with pairwise possible-discernibility of the modes.

We will now embed the problem of detecting topological changes in a network of DAEs as discussed above in the framework of switched DAEs. Therefore consider a family of (possibly non-regular) DAEs (1) connected by a graph $\mathfrak{G} = (\mathfrak{V}, \mathfrak{E})$ with Laplacian $\mathfrak{L}$ and corresponding matrix pair $(\mathcal{E}_0, \mathcal{A}_0)$ given by
$$\mathcal{E}_0 = \mathcal{E} \quad \text{and} \quad \mathcal{A}_0 = \mathcal{A} - \mathcal{B}\mathfrak{L}\mathcal{C} \qquad (14)$$
as in (3). Let $\mathfrak{L}_1, \mathfrak{L}_2, \ldots, \mathfrak{L}_K$, $K \in \mathbb{N}$, denote the Laplacian matrices resulting from all relevant (in particular, regularity preserving) removal/addition of each of the edges $(i_1, j_1), \ldots, (i_K, j_K)$ in $\mathfrak{E}$ and
$$\mathcal{E}_k = \mathcal{E} \quad \text{and} \quad \mathcal{A}_k = \mathcal{A} - \mathcal{B}\mathfrak{L}_k\mathcal{C}. \qquad (15)$$
Now we are interested in switching signals σ with $\sigma(t) = 0$ for $t \leq 0$ and with at most one switch. In this case, t_S-observability is equivalent to mode 0 being discernible from any other mode $k \in \{1, \ldots, K\}$. The condition for switching signal observability remains unaffected.

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