Solvability and stability of a power system DAE model

Tjorben Groß, Stephan Trenn[a], Andreas Wirsen[b]

[a] *FB Mathematik, TU Kaiserslautern, Germany*
[b] *Fraunhofer ITWM, Kaiserslautern, Germany*

Abstract

The dynamic model of a power system is the combination of the power flow equations and the dynamic description of the generators (the swing equations) resulting in a differential-algebraic equation (DAE). For general DAEs solvability is not guaranteed in general, in the linear case the coefficient matrices have to satisfy a certain regularity condition. We derive a solvability characterization for the linearized power system DAE solely in terms of the network topology. As an extension to previous result we allow for higher order generator dynamics. Furthermore, we show that any solvable power system DAE is automatically of index one, which means that it is also numerically well posed. Finally, we show that any solvable power system DAE is stable but not asymptotically stable.

Keywords: Power Systems, DAE, Solvability, Stability

1. Introduction

The modeling, analysis and simulation of the power grid is of high importance to ensure reliable and efficient power generation now and in the future. A common dynamical model of power grids on the transmission level is a description with the so called swing equations (linear or nonlinear ODEs) for the generators in combination with the power flow equations (nonlinear algebraic equations), see e.g. [1]. This combination results in a nonlinear differential-algebraic equation (DAE). This nonlinear DAE model was only studied by a few authors, e.g. in the context of bifurcation [2, 3, 4], observer design [5, 6], pseudospectra analysis [7] and cyber-physical security [8]. There are no general results available for the solvability and stability of nonlinear DAEs, therefore, one often considers a linearized model. Even in the linear case existence and uniqueness of solutions of general DAEs is not trivial (see e.g. [9]) and we have investigated these issues in [10].

Our main contribution in this note is the analysis of a linear DAE model of power grids which takes into account *higher order generator models*, i.e. instead of modeling a generator with just one rotating mass, we model a generator (together with the turbines) as several coupled rotating masses. This more detailed generator model is for example necessary to better understand effects like sub synchronous resonances [11]. For this more sophisticated DAE model we generalize the solvability characterization from [10], i.e. we show that existence and uniqueness of solutions (well-posedness) can be checked by a simple topological condition of the connectivity of the network graph. For the first time, we also derive a stability characterization for this linear DAE model. We show that any well-posed linear power grid DAE model is stable, but it is *not asymptotically stable*. As a consequence small nonlinear disturbance can make the whole power grid DAE unstable and therefore we claim that it might be dangerous to rely on the linear model when studying stability of power grids.

It is common to simplify the original power grid DAE model to an ordinary differential equation (ODE) model by resolving the algebraic constraints, however this is only feasible when the DAE model has index one. We show that this is indeed the case when the power grid does not contain disconnected loads, so one may wonder what the advantages of a DAE formulation is. First of all, the modeling as a DAE is much simpler as it is not necessary to solve the algebraic constraints; in particular, when using automated modeling tools one usually obtains a DAE. Secondly, when considering the possibility of faults leading to sudden structural or topological changes (e.g. the disconnection of power lines) the underlying algebraic constraints change; hence the resulting ODEs are not "compatible" anymore with each other. Although, we do not investigate these issues here, our results presented in a DAE framework will be a starting point for future research concerning the analysis and control of power grids in the presence of sudden structural changes.

2. Derivation of power system DAE

We consider a power network consisting of $n \in \mathbb{N}$ generators (connected to n generator buses) and $m \in \mathbb{N}$ load buses interconnected by transmission lines. The transmission lines are described by Π-models (see [12, 1]). The electrical interconnections of the generators with the power grid are represented by constant voltage behind transient reactance models (see [13, 1]) and the mechanical interconnection of the i-th generator with driving turbines is modeled with $\eta_i \in \mathbb{N}$ rotating masses as illustrated in Figure 1.

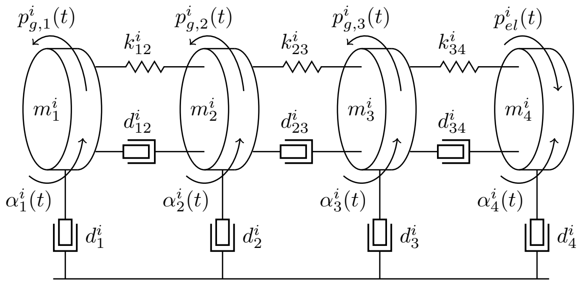

Figure 1: A multi-mass model for the i-th generator with $\eta_i = 4$ masses.

The linear differential equations representing the dynamic behavior of the i-th generator are

$$\begin{aligned} \dot{\alpha}^i(t) &= \omega^i(t) \\ M^i \dot{\omega}^i(t) &= -D^i \omega^i(t) - K^i \alpha^i(t) + P^i_g(t) - P^i_e(t), \end{aligned}$$

where, omitting the time dependency, $\alpha^i = (\alpha^i_1, \ldots, \alpha^i_{\eta_i})^\top \in \mathbb{R}^{\eta_i}$ and $\omega^i = (\omega^i_1, \ldots, \omega^i_{\eta_i})$ are the angles and the angular velocities[1] of the η_i rotating masses, $P^i_g = (p^i_{g,1}, \ldots, p^i_{g,\eta_i})^\top \in \mathbb{R}^{\eta_i}$ is the generator power acting on the rotating masses (the turbines[2]) and $P^i_e = (0, \ldots, 0, p^i_e) \in \mathbb{R}^{\eta_i}$ is the electrical power acting in the opposite direction on the last rotating mass (the actual generator). Furthermore the matrices $M^i, D^i, K^i \in \mathbb{R}^{\eta_i \times \eta_i}$ have the following structures:

$$M^i = \begin{bmatrix} m^i_1 & & & \\ & m^i_2 & & \\ & & \ddots & \\ & & & m^i_{\eta_i} \end{bmatrix}$$

$$D^i = \begin{bmatrix} d^i_{11}+d^i_{12} & -d^i_{12} & & & \\ -d^1_{12} & d^i_{22}+d^i_{12}+d^i_{23} & -d^i_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & -d^i_{\eta_i-2,\eta_i-1} & d^i_{\eta_i-1,\eta_i-1}+d^i_{\eta_i-2,\eta_i-1}+d^i_{\eta_i-1,\eta_i} & -d^i_{\eta_i-1,\eta_i} \\ & & & -d^i_{\eta_i-1,\eta_i} & d^i_{\eta_i,\eta_i}+d^i_{\eta_i-1,\eta_i} \end{bmatrix}$$

$$K^i = \begin{bmatrix} k^i_{12} & -k^i_{12} & & & \\ -k^1_{12} & k^i_{12}+k^i_{23} & -k^i_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & -k^i_{\eta_i-2,\eta_i-1} & k^i_{\eta_i-2,\eta_i-1}+k^i_{\eta_i-1,\eta_i} & -k^i_{\eta_i-1,\eta_i} \\ & & & -k^i_{\eta_i-1,\eta_i} & k^i_{\eta_i-1,\eta_i} \end{bmatrix}$$

with positive entries $m^i_p, d^i_{pq}, k^i_{pq}$ for corresponding indexes $p, q \in \{1, 2 \ldots, \eta_i\}$. The electrical power p^i_e acting on the i-th generator is modeled via the constant voltage behind transient reactance assumption [14] in terms of the generator angle $\alpha^i_{\eta_i}$ and generator bus voltage angle θ_i. In particular, we assume that the mathematical expression for the electrical power derived for constant $\alpha^i_{\eta_i}$ and θ_i (i.e. when the dynamical system is in steady state) is also valid for slowly time varying $\alpha^i_{\eta_i}$ and θ_i. Assuming furthermore (c.f.[8, 10]) that the difference between $\alpha^i_{\eta_i}$ and θ^i is small and that the generator bus voltage amplitude is locally regulated to be constant (and given in per unit), we can express p^i_e linearly as follows

$$p^i_e(t) = \frac{1}{z_i}(\alpha^i_{\eta_i}(t) - \theta^i(t))$$

where $z^i > 0$ is the transient reactance of the generator. Assuming furthermore that the bus voltage angle differences $\theta^i - \theta^j$, $i, j = 1, 2, \ldots, n+m$ are small and that the conductances between the buses are negligible, we can linearize the power flow equations (c.f. [10]) to obtain the linear constraints

$$\begin{aligned} p^i(t) + p^i_e(t) &= \sum_{j=1}^{n+m} b_{ij}(\theta^i(t) - \theta^j(t)), \\ &\text{for } i = 1, \ldots, n, \\ p^i(t) &= \sum_{j=1}^{n+m} b_{ij}(\theta^i(t) - \theta^j(t)), \\ &\text{for } i = n+1, \ldots, n+m, \end{aligned}$$

[1] In fact, here ω_i denotes the relative deviation from a constant nominal angular velocity.

[2] For $\eta_i > 1$ the last rotating mass is usually not a turbine, i.e. $p^i_{g,\eta_i} = 0$.

where $b_{ij} = b_{ji} \geq 0$ is the susceptance between bus i and bus j and $p^i(t)$ is the active power infeed at the i-th bus representing time-dependent loads. Note that $b_{ij} = 0$ means that bus i and j are not directly connected, in particular the matrix $B = [b_{ij}]_{i,j=1,\dots,n+m}$ is a weighted adjacency matrix of the power network graph. Altogether this model results in the following linear network DAE:

$$\mathfrak{E}\dot{x} = \mathfrak{A}x + \mathfrak{B}u \tag{1}$$

with variables

$$x := \begin{pmatrix} \alpha \\ \omega \\ \widehat{\theta} \\ \widetilde{\theta} \end{pmatrix} \in \mathbb{R}^{n_\eta+n_\eta+n+m}, \quad n_\eta := \sum_{i=1}^{n} \eta_i$$

$$u := \begin{pmatrix} P_g \\ \widehat{P} \\ \widetilde{P} \end{pmatrix} \in \mathbb{R}^{n_\eta+n+m},$$

$$\alpha := \left(\alpha_1^1,\dots,\alpha_{\eta_1}^1,\alpha_1^2,\dots,\alpha_{\eta_2}^2,\dots,\alpha_1^n,\dots,\alpha_{\eta_n}^n \right)^\top \in \mathbb{R}^{n_\eta},$$

$$\omega := \left(\omega_1^1,\dots,\omega_{\eta_1}^1,\omega_1^2,\dots,\omega_{\eta_2}^2,\dots,\omega_1^n,\dots,\omega_{\eta_n}^n \right)^\top \in \mathbb{R}^{n_\eta},$$

$$\widehat{\theta} := \left(\theta_1,\dots,\theta_n \right)^\top \in \mathbb{R}^n, \ \widetilde{\theta} := \left(\theta_{n+1},\dots,\theta_{n+m} \right)^\top \in \mathbb{R}^m,$$

$$P_g := \left(P_g^{1\top},\dots,P_g^{n\top} \right)^\top \in \mathbb{R}^{n_\eta},$$

$$\widehat{P} := \left(p_1,\dots,p_n \right)^\top \in \mathbb{R}^n, \ \widetilde{P} := \left(p_{n+1},\dots,p_{n+m} \right)^\top \in \mathbb{R}^m,$$

and matrices

$$\mathfrak{E} = \begin{bmatrix} I_{n_\eta} & 0 & 0 & 0 \\ 0 & M & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathfrak{A} = \begin{bmatrix} 0 & I_{n_\eta} & 0 & 0 \\ -K-HZ^{-1}H^\top & -D & HZ^{-1} & 0 \\ Z^{-1}H^\top & 0 & -R_1-Z^{-1} & -R_2 \\ 0 & 0 & -R_3 & -R_4 \end{bmatrix},$$

$$\mathfrak{B} = \begin{bmatrix} 0 & 0 & 0 \\ I_{n_\eta} & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_m \end{bmatrix},$$

with the block entries $M := \operatorname{diag}(M^1, \dots, M^n)$, $D := \operatorname{diag}(D^1, \dots, D^n)$, $K := \operatorname{diag}(K^1, \dots, K^n)$, $Z := \operatorname{diag}(z^1, \dots, z^n)$,

$$H := \begin{bmatrix} H^1 \\ \vdots \\ H^n \end{bmatrix} \in \mathbb{R}^{n_\eta \times n}, \quad H^i := \begin{bmatrix} 0 \\ e_i^\top \end{bmatrix} \in \mathbb{R}^{\eta_i \times n},$$

$e_i \in \mathbb{R}^n$ is the i-th unit vector and $R_1 \in \mathbb{R}^{n\times n}$, $R_4 \in \mathbb{R}^{m\times m}$, $R_2 = R_3^\top \in \mathbb{R}^{n\times m}$ are such that

$$\begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix} := R := -B + \operatorname{diag}\left(\sum_{j=1}^{n+m} b_{1k}, \dots, \sum_{j=1}^{n+m} b_{n+m,k} \right), \tag{2}$$

i.e. R is the (weighted) Laplacian of the graph describing the connectivity of the power grid.

The following properties of K, D and R given above are a simple consequence from Gershgorin's Circle Theorem.

Remark 2.1.

1. *The symmetric matrices K and R are positive semidefinite.*
2. *The symmetric matrix D is positive definite.*

3. Regularity of linearized model

Definition 3.1 (Regularity)**.** *A matrix pair (E, A) with square matrices $E, A \in \mathbb{R}^{k\times k}$, $k \in \mathbb{N}$, is called* regular *if, and only if,* $\det(Es - A)$ *is not the zero polynomial. Furthermore, we call a DAE $E\dot{x} = Ax + Bu$ regular when the corresponding matrix pair (E, A) is regular.*

It is well known (c.f. [15]) that regularity of a linear DAE is necessary and sufficient for existence and uniqueness of solutions. Hence characterizing regularity for the linearized power system model is crucial; without regularity it is not possible to run simulations or make stability statements.

Fortunately, there is a surprisingly simple and also very intuitive characterization for regularity solely in terms of the power network's graph topology.

Theorem 3.2 (Regularity characterization of power network DAE)**.** *Consider the linearized DAE model (1) for a power system with n generators and m load buses. Then this DAE is regular (i.e. existence and uniqueness of solutions is guaranteed) if, and only if, every load bus is connected via a path in the network graph (given by the adjacency matrix B) to a generator bus. In particular, if the whole power network graph is connected, then existence and uniqueness of solutions is guaranteed.*

Before presenting the proof we state the following key lemma.

Lemma 3.3. *For a power network with symmetric susceptance matrix $B \in \mathbb{R}^{(n+m)\times(n+m)}$, let $R \in \mathbb{R}^{(n+m)\times(n+m)}$ be given as in (2). Then the following statements are equivalent:*

1. *Every load bus is connected (via a path in the network graph) with a generator.*
2. *There exists a positive definite $Q = Q^\top \in \mathbb{R}^{n\times n}$ such that*

$$R + \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \text{ is positive definite.} \tag{3}$$

3. *For all positive definite $Q = Q^\top \in \mathbb{R}^{n\times n}$ (3) holds.*

Proof. Invoking Corollary 7.2 from the Appendix we see that $x^\top Rx \geq 0$ for all $x \in \mathbb{R}^{n+m}$ and $x^\top Rx = 0$ if, and only if $x_i = x_j$ for all (i,j) with $b_{ij} \neq 0$. In particular, $x^\top Rx = 0$ implies $x_i = x_j$ for all buses i, j which are connected via a path in the network graph.
$1 \Rightarrow 3$. By positive semidefiniteness of R we have for all $x \in \mathbb{R}^{n+m}$ and all positive definite Q that

$$x^\top \left(R + \left[\begin{smallmatrix} Q & 0 \\ 0 & 0\end{smallmatrix}\right]\right) x = 0 \quad \Leftrightarrow \quad x^\top Rx = 0 \ \wedge \ x^\top \left[\begin{smallmatrix} Q & 0 \\ 0 & 0\end{smallmatrix}\right] x = 0.$$

By assumption every load bus $j \in \{n+1, n+2, \ldots, n+m\}$ is connected to some generator bus $i \in \{1, 2, \ldots, n\}$ and therefore $x^\top Rx = 0$ implies $x_j = x_i$ and $x^\top \left[\begin{smallmatrix} Q & 0 \\ 0 & 0\end{smallmatrix}\right] x = 0$ implies $x_i = 0$. Hence $x^\top \left(R + \left[\begin{smallmatrix} Q & 0 \\ 0 & 0\end{smallmatrix}\right]\right) x = 0$ implies $x = 0$, i.e. the positive semidefinite matrix $R + \left[\begin{smallmatrix} Q & 0 \\ 0 & 0\end{smallmatrix}\right]$ is in fact positive definite.
$3 \Rightarrow 2$. Trivially true.
$2 \Rightarrow 1$. Seeking a contradiction assume there is a load bus $j^* \in \{n+1, \ldots, n+m\}$ which is not connected with any generator bus. Define $x^* \in \mathbb{R}^{n+m}$ as follows: For $j \in \{1, 2, \ldots, n+m\}$ let $x_j^* = 1$ if load bus j is connected via a path with node j^* and $x_j^* = 0$ otherwise, in particular $x_i^* = 0$ for all generator buses $i \in \{1, 2, \ldots, n\}$. By construction, ${x^*}^\top Rx^* = 0$ and ${x^*}^\top \left[\begin{smallmatrix} Q & 0 \\ 0 & 0\end{smallmatrix}\right] x^* = 0$, i.e. $R + \left[\begin{smallmatrix} Q & 0 \\ 0 & 0\end{smallmatrix}\right]$ is not negative definite. $\square$

Proof of Theorem 3.2. Step 1: We show that $(\mathfrak{E}, \mathfrak{A})$ is regular if, and only if, $\det(L(s))$ is not identically zero, where

$$L(s) := R + \left[\begin{smallmatrix} Z^{-1} - Z^{-1}H^\top\left(Ms^2 + Ds + K + HZ^{-1}H^\top\right)^{-1}HZ^{-1} & 0 \\ 0 & 0\end{smallmatrix}\right].$$

First note, that M is a diagonal matrix with non-zero diagonal entries, hence with

$$W(s) := s^2M + sD + K + HZ^{-1}H^\top$$

we have that $\det(W(s))$ is a non-zero polynomial (of degree $2n_\eta$) and hence $L(s)$ is indeed well-defined as a rational matrix. Furthermore, with

$$Q(s) := \left[\begin{smallmatrix} I_{n_\eta} & & & \\ Ms+D & I_{n_\eta} & & \\ Z^{-1}H^\top W(s)^{-1}(Ms+D) & Z^{-1}H^\top W(s)^{-1} & I_n & \\ & & & I_m \end{smallmatrix}\right]$$

we have

$$Q(s)(\mathfrak{E}s - \mathfrak{A}) = \left[\begin{smallmatrix} \left[\begin{smallmatrix} sI_{n_\eta} & -I_{n_\eta} \\ W(s) & 0 \end{smallmatrix}\right] & \left[\begin{smallmatrix} 0 & 0 \\ -HZ^{-1} & 0\end{smallmatrix}\right] \\ 0 & L(s) \end{smallmatrix}\right].$$

Since $\det(Q(s)) = 1$ we have

$$\det(\mathfrak{E}s - \mathfrak{A}) = \det(W(s))\det(L(s))$$

and, recalling that $\det(W(s))$ is not the zero polynomial, the claim of Step 1 is shown.
Step 2: We show "$\Leftarrow$".

By assumption Z^{-1} is positive definite. Furthermore, $\det(W(s))$ is a polynomial with degree $2n_\eta > 0$, therefore we have that the symmetric matrix $Z^{-1}H^\top W(\lambda)^{-1}HZ^{-1}$ converges to zero for $\lambda \to \infty$. Hence for sufficiently large $\lambda \in \mathbb{R}$ we have that $Z^{-1} - Z^{-1}H^\top W(\lambda)^{-1}HZ^{-1}$ is positive definite and by Lemma 3.3 we have that $L(\lambda)$ is invertible. In particular, $\det(L(s))$ is not identically zero and by Step 1 we conclude that $(\mathfrak{E}, \mathfrak{A})$ is regular.
Step 3: We show "$\Rightarrow$".

By Step 1 we can assume that the rational function $\det(L(s))$ is not identically zero, hence $L(\lambda)$ is an invertible matrix for almost all $\lambda \in \mathbb{R}$. Similar as in Step 2 we can choose $\lambda \in \mathbb{R}$ sufficiently large so that the symmetric matrix $Q := Z^{-1} - Z^{-1}H^\top W(\lambda)^{-1}HZ^{-1}$ is positive definite and $L(\lambda)$ is invertible. Now invertibility of $L(\lambda) = R + \left[\begin{smallmatrix} Q & 0 \\ 0 & 0\end{smallmatrix}\right]$ implies by Lemma 3.3 that every load bus is connected to a generator.

$\square$

Remark 3.4 (Invertibility of R_4)**.** *Positive definiteness of $\left[\begin{smallmatrix} R_1+Q & R_2 \\ R_3 & R_4\end{smallmatrix}\right]$ implies positive definiteness of R_4. In view of Theorem 3.2 together with Lemma 3.3 we can therefore conclude that regularity of the power system DAE implies invertibility of the submatrix R_4 in (2).*

4. Index one

For numerical simulations of a DAE the so called index plays a crucial role. Roughly speaking the higher the index, the more difficult it is to run numerical simulations. The index of a regular DAE is most conveniently defined in terms of the quasi-Weierstrass form:

Lemma 4.1 (Quasi-Weierstrass form (QWF), c.f. [16])**.** *A square matrix pair (E, A) is regular if, and only if, there exists invertible matrices S, T such that*

$$(SET, SAT) = \left(\left[\begin{smallmatrix} I & 0 \\ 0 & N\end{smallmatrix}\right], \left[\begin{smallmatrix} J & 0 \\ 0 & I\end{smallmatrix}\right]\right) \tag{4}$$

where N is nilpotent.

Definition 4.2 (Index of a linear DAE)**.** *The index of the regular matrix pair (E, A) (or the corresponding DAE) is the nilpotency index of N in the QWF (4), i.e. the minimal number $\nu \in \mathbb{N}$ such that $N^\nu = 0$.*

For a DAE with index larger than one, it is possible that derivatives of the input occur in the solution and inconsistent initial values lead to Dirac impulses (and their derivatives), see e.g. [9]; both effects may lead to numerical instabilities. For the index-one case these problems do not occur and the DAE is numerically solvable. We will show now that the network graph topological condition ensuring regularity (i.e. solvability of the DAE) already implies index-one (i.e. numerical solvability).

Theorem 4.3 (Index one)**.** *Consider the linearized power system* (1)*. If* (1) *is regular, then it has index one.*

Proof. Regularity of (1) implies by Theorem 3.2 and Lemma 3.3 that

$$A_4 := -R - \begin{bmatrix} Z^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

is invertible. Since $(\mathfrak{E}, \mathfrak{A})$ can be written as

$$(\mathfrak{E}, \mathfrak{A}) = \left(\begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \right)$$

with invertible E_1 and A_4, it is easily seen that

$$S = \begin{bmatrix} E^{-1} & -E^{-1}A_2A_4^{-1} \\ 0 & A_4^{-1} \end{bmatrix} \text{ and } T = \begin{bmatrix} I & 0 \\ -A_4^{-1}A_3 & I \end{bmatrix}$$

transform $(\mathfrak{E}, \mathfrak{A})$ into the QWF

$$(S\mathfrak{E}T, S\mathfrak{A}T) = \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} E_1^{-1}(A_1 - A_2A_4^{-1}A_3) & 0 \\ 0 & I \end{bmatrix} \right).$$

This shows the index one property (because $N = 0$). ☐

5. Stability

Definition 5.1 (Stability)**.** *A DAE $E\dot{x} = Ax + Bu$ is called* stable*, if and only if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any solution x of the homogeneous DAE $E\dot{x} = Ax$ the following implication holds:*

$$\|x(0)\| < \delta \quad \Rightarrow \quad \forall t \geq 0 : \|x(t)\| < \varepsilon.$$

A DAE is called asymptotically stable*, if and only if, the DAE is stable and additionally any solution of the homogenouos DAE converges to zero.*

For DAEs with square coefficient matrices, the following well known characterization of stability holds:

Lemma 5.2. *The DAE $E\dot{x} = Ax + Bu$ with square coefficient matrices (E, A) is*

- *asymptotically stable if, and only if,*

$$\operatorname{spec}(Es - A) := \{ \lambda \in \mathbb{C} \mid \det(\lambda E - A) = 0 \} \subseteq \mathbb{C}_{\mathrm{Re}<0},$$

i.e. all generalized eigenvalues of the pair (E, A) have negative real part.

- *stable, if and only if,*

$$\operatorname{spec}(Es - A) \subseteq \mathbb{C}_{\mathrm{Re}\leq 0}$$

and for all $\lambda \in \operatorname{spec}(Es - A) \cap i\mathbb{R}$ the geometric and algebraic multiplicities of λ are equal, i.e.

$$\dim \ker_{\mathbb{C}}(\lambda E - A) = \max \left\{ k \in \mathbb{N} \;\middle|\; \tfrac{\det(Es - A)}{(s-\lambda)^k} \text{ is a polynomial} \right\}.$$

Note that in the square case non-regularity of (E, A) implies $\operatorname{spec}(Es - A) = \mathbb{C}$, hence regularity is a necessary assumption for (asymptotic) stability. Therefore, the above stability characterization is actually a simple consequence of the quasi-Weierstrass form (4), because the DAE $E\dot{x} = Ax$ is (asymptotically) stable if, and only if, $\dot{v} = Jv$ is (asymptotically) stable. Furthermore, for any $\lambda \in \mathbb{C}$ it holds that

$$\det(\lambda E - A) = 0 \quad \Leftrightarrow \quad \det(\lambda I - J) = 0.$$

We can now state the main stability result. Similar to the index-one result, regularity (i.e. unique solvability) of the linearized power system DAE already implies stability. However, asymptotic stability does not hold, i.e. certain initial states will not converge to zero as time goes to infinity.

Theorem 5.3 (Stability)**.** *Consider the linearized power system DAE* (1)*. If* (1) *is regular, then it is stable but not asymptotically stable.*

Proof. It suffices to show the corresponding stability property for the ODE part in the quasi-Weierstrass form (4). Following the lines of the proof of Theorem 4.3, we see that J in the quasi-Weierstrass form of $(\mathfrak{E}, \mathfrak{A})$ is given by

$$J = \begin{bmatrix} 0 & I \\ -M^{-1}(K + HZ^{-1}H^\top + HZ^{-1}UZ^{-1}H^\top) & -M^{-1}D \end{bmatrix}$$

where $U \in \mathbb{R}^{n\times n}$ is such that

$$\left[\begin{smallmatrix} U & * \\ * & * \end{smallmatrix}\right] = \left(-R - \left[\begin{smallmatrix} Z^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right]\right)^{-1},$$

i.e. (invoking Schur's complement) $U = (-R_1 - Z^{-1} + R_2R_4^{-1}R_3)^{-1}$. Note that symmetry of R implies symmetry of U.

Step 1: We show that all non-zero generalized eigenvalues have negative real parts.

Step 1a: We show $Z^{-1} + Z^{-1}UZ^{-1}$ is positive semidefinite.
First note that $R + \left[\begin{smallmatrix} Z^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is a so-called M-matrix because it has non-positive off-diagonal entries and is positive definite. It is a well known fact of M-matrics (see e.g. [17]) that the inverse of an M-matrix has non-negative entries. Hence U as the negative of an upper left block of the inverse of an M-matrix has non-positive entries. Since Z^{-1} is a positive diagonal matrix, $Z^{-1} + Z^{-1}UZ^{-1}$ has non-positive off-diagonal entries. Invoking Corollary 7.2 from the Appendix it remains to show that $(1,\dots,1)^\top$ is in the kernel of $Z^{-1}+Z^{-1}UZ^{-1}$. Multiplying the latter from the left with $-U^{-1}Z$ we just have to show that

$$(R_1 - R_2R_4^{-1}R_3)\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} = 0.$$

But this is a simple consequence from the fact that $R(1,\dots,1)^\top = 0$.

Step 1b: Reformulation as quadratic eigenvalue problem.
From Step 1a we can conclude that $HZ^{-1}H^\top + HZ^{-1}UZ^{-1}H^\top$ is positive semidefinite. Hence

$$\widehat{K} := K + HZ^{-1}H^\top + HZ^{-1}UZ^{-1}H^\top$$

is also positive semidefinite. It is now easily seen that $\lambda \in \mathbb{C}$ is an eigenvalue of J if, and only if, λ solves the quadratic eigenvalue problem

$$\det(\lambda^2 M + \lambda D + \widehat{K}) = 0. \tag{5}$$

It is well known (see e.g. [18]) that symmetry and positive definiteness of M and positive semidefiniteness of D and $\widehat{K}$ implies that all solutions $\lambda \in \mathbb{C}$ of (5) satisfy $\mathrm{Re}(\lambda) \le 0$. Furthermore, if $\lambda \in \mathbb{C}$ solves (5) then there exists $p \in \mathbb{C}^{n_\nu} \setminus \{0\}$ such that

$$\underbrace{\overline{p}^\top Mp}_{\in\mathbb{R}}\lambda^2 + \underbrace{\overline{p}^\top Dp}_{\in\mathbb{R}\setminus\{0\}}\lambda + \underbrace{\overline{p}^\top \widehat{K}p}_{\in\mathbb{R}} = 0,$$

hence λ cannot be purely imaginary which shows that all the solutions λ of (5) are either zero or have negative real part.

Step 2: We show that zero is an eigenvalue and derive the corresponding eigenspace.
First observe that for $x, y \in \mathbb{R}^{n_\eta}$ the following equivalence holds:

$$J\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = 0 \quad\Leftrightarrow\quad y = 0 \ \wedge\ \widehat{K}x = 0.$$

We will now show that

$$\ker \widehat{K} = \operatorname{span}\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}. \tag{6}$$

Taking into account the block-diagonal structure of K and $HZ^{-1}H^\top$ and Lemma 7.3 in the Appendix, we can conclude that $K + HZ^{-1}H^\top$ is invertible and

$$\widehat{H} := (K + HZ^{-1}H^\top)^{-1}HZ^{-1} = \begin{pmatrix} \begin{smallmatrix}1\\ \vdots\\ 1\end{smallmatrix} & & & \\ & \begin{smallmatrix}1\\ \vdots\\ 1\end{smallmatrix} & & \\ & & \ddots & \\ & & & \begin{smallmatrix}1\\ \vdots\\ 1\end{smallmatrix} \end{pmatrix} \in \mathbb{R}^{n_\eta \times n}.$$

Then, by definition, the following equivalence holds:

$$\widehat{K}x = 0 \quad\Leftrightarrow\quad x + \widehat{H}UZ^{-1}H^\top x = 0.$$

Let $\widehat{\eta}_i := \sum_{k=1}^{i} \eta_i$ and for any $x \in \ker \widehat{K}$ let $x^* := H^\top x = (x_{\widehat{\eta}_1}, x_{\widehat{\eta}_2}, \dots, x_{\widehat{\eta}_n})^\top$, then

$$x_j + (UZ^{-1})_i x^* = 0 \qquad \begin{array}{l}\forall i \in \{1,\dots,n\},\\ \forall j \in \{\widehat{\eta}_{i-1}+1,\dots,\widehat{\eta}_i\},\end{array}$$

where $(UZ^{-1})_i$ denotes the i-th row of UZ^{-1}. In particular,

$$x_j = x_{\widehat{\eta}_i} = x_i^* \qquad \begin{array}{l}\forall i \in \{1,\dots,n\},\\ \forall j \in \{\widehat{\eta}_{i-1}+1,\dots,\widehat{\eta}_i\},\end{array} \tag{7}$$

and therefore

$$\begin{aligned} x^* + UZ^{-1}x^* = 0 &\quad \text{or, equivalently,} \\ U^{-1}x^* + Z^{-1}x^* = 0. & \end{aligned}$$

Recalling that $U^{-1} = -R_1 - Z^{-1} + R_2R_4^{-1}R_3$ we conclude that

$$(R_1 - R_2R_4^{-1}R_3)x^* = 0.$$

Since we assumed that the power grid is connected it follows that $\ker R = \operatorname{span}(1,\dots,1)^\top$ and hence

$x^* \in \operatorname{span}(1,\ldots,1)^\top$. From (7) we can now conclude (6).

Step 3: We show that the zero eigenvalue has coinciding algebraic and geometric multipliciy.
The eigenvalue zero of J has equal algebraic and geometric multiplicity if, and only if, there are no non-zero nilpotent Jordan blocks in the Jordan canonical form, i.e.

$$\ker J = \ker J^2.$$

It is easily seen that

$$\begin{aligned}
&J^2 \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = 0 \\
\Leftrightarrow \quad & y = -(M^{-1}D)^{-1}M^{-1}\widehat{K}x \wedge \widehat{K}D^{-1}\widehat{K}x = 0.
\end{aligned}$$

Hence it suffices to show that

$$\ker \widehat{K} = \ker \widehat{K}D^{-1}\widehat{K}. \tag{8}$$

or, equivalently,

$$\operatorname{im} D^{-1}\widehat{K} \cap \ker \widehat{K} = \{0\} \tag{9}$$

Due to symmetry of the involved matrices, it holds that

$$\operatorname{im} D^{-1}\widehat{K} = (\ker \widehat{K}D^{-1})^\perp = (D \ker \widehat{K})^\perp.$$

From Step 2 and the definition of D we see that $D \ker \widehat{K}$ is spanned by the vector consisting of the positive diagonal entries of D. In particular, all non-zero vectors in the orthogonal complement of $D \ker \widehat{K}$ must have entries with both positive and negative signs. Since all entries of the vectors in $\ker \widehat{K}$ have identical entries (and in particular identical signs), we have shown (9). ☐

Remark 5.4 (Existence of a Lyapunov function)**.** *It can be shown (cf. [19, 4.8]) that for any stable index-1 DAE there exists a pair of matrices (P,Q) such that*

$$A^\top PE + E^\top PA = -E^\top QE, \quad Q \geq 0$$

and $V(x) = x^\top E^\top PEx$ is a Lyapunov function

6. Conclusion

We have derived a linearized DAE model for a power system with generators consisting of multiple coupled rotating masses. Utilizing the special structure of the resulting linear DAE we have characterized regularity, index one and stability.

The extension of this results to the nonlinear case is still an open topic; in particular, since the linearized model is not asymptotically stable Lyapunov's Indirect Method *cannot* be applied to conclude that the nonlinear model is at least locally stable. Further research with novel approaches are necessary to establish stability of the nonlinear DAE model.

Sudden structural changes (like disconnection of lines or deactivation of generators) result in a switched DAE model and stability of each mode does not guarantee stability of the switched system, it is therefore necessary to investigate the effect of switches on stability of power systems and will be a topic of future research.

7. Appendix

Lemma 7.1. *Consider a symmetric matrix $W = [w_{ij}]_{i,j=1,\ldots,n} \in \mathbb{R}^{n\times n}$ with $W(1,1,\ldots,1)^\top = 0$, i.e. each diagonal element is the negative sum of the off-diagonal row elements. Then, for all $x \in \mathbb{R}^n$,*

$$x^\top Wx = -\sum_{i=2}^{n}\sum_{j=1}^{i-1} w_{ij}(x_i - x_j)^2.$$

Proof. This is a straight-forward calculation (c.f. the first part of the proof of [10, Lem. 2]). ☐

Corollary 7.2. *Let $W \in \mathbb{R}^{n\times n}$ be a symmetric matrix with $W(1,1,\ldots,1)^\top = 0$ and non-positive off-diagonal entries, i.e. W is a (weighted) Laplacian of some graph. Then the following statements hold:*

1. *W is positive semidefinite.*
2. *Let $\mathscr{G}_W = (\mathscr{V}, \mathscr{E})$ be the graph induced by W, i.e. $\mathscr{V} = \{1,\ldots,n\}$ and $(i,j) \in \mathscr{E}$ if, and only if, $w_{ij} \neq 0$, then for any $x \in \mathbb{R}^n$*
$$x^\top Wx = 0 \quad \Leftrightarrow \quad \forall (i,j) \in \mathscr{E}: \; x_i = x_j.$$
3. *$\mathscr{G}_W$ is connected if, and only if,*
$$\ker W = \operatorname{span} \left(\begin{smallmatrix} 1 \\ \vdots \\ 1 \end{smallmatrix}\right).$$

Lemma 7.3. *Let $K \in \mathbb{R}^{n\times n}$ be a symmetric tridiagonal matrix with positive off-diagonal entries and with $\ker K = \operatorname{span}(1,1,\ldots,1)^\top$ (i.e. the diagonal is the negative sum of the off-diagonal elements next to it). Then for any $z \neq 0$ the matrix*

$$K + \left[\begin{smallmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & z \end{smallmatrix}\right]$$

is invertible and the last column of the inverse is $(1/z, 1/z, \ldots, 1/z)^\top$.

Proof. This is easily seen by carrying out Gauß-eliminations. $\square$

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