# The Funnel Controller: More than a decade of adaptation

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Technische Universität Kaiserslautern

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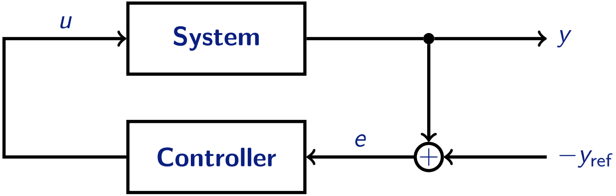
# Control Task

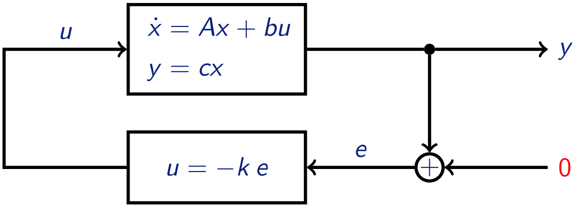



Goal: Output tracking

without

- exact knowledge of system model
- knowledge of reference signal (only error $e$ is available for controller)
- asymptotic convergence (but arbitrarily small error)

# High-gain-Feedback: linear case



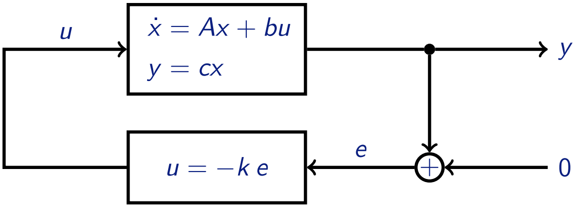

Assumptions:

- Relative degree $r = 1$ $\Leftrightarrow$ $\gamma := cb \neq 0$ , in particular:

$$\text{System} \quad \Leftrightarrow \quad \begin{aligned} \dot{y} &= a_{11}y + a_{12}z + \gamma u \\ \dot{z} &= a_{21}y + A_{22}z \end{aligned}$$

- positive high frequency gain $\Leftrightarrow$ $\gamma > 0$
- stable zero-dynamics (minimum phase) $\Leftrightarrow$ $A_{22}$ Hurwitz

# High-gain-Feedback: linear case



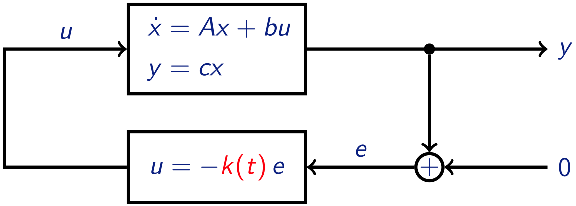

**Theorem (High-Gain Feedback)**

*$cb > 0$ and stable zero-dynamics*

$$\Rightarrow \quad \exists k_0 > 0\ \forall\, k \geq k_0 : \ \textit{Closed loop is asymptotically stable}$$

Problem: How to find $k_0$?

# High-gain-Feedback: linear case




Idea: Make gain $k$ time varying

**Theorem (Adaptive High-Gain Feedback,** Byrnes & Willems 1984**)**

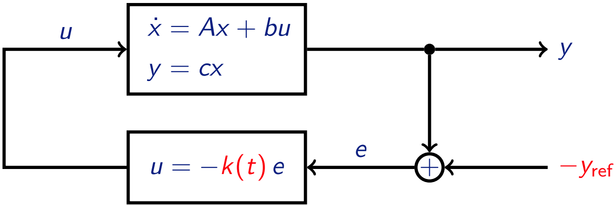
*$cb > 0$ and stable zero-dynamics* $\Rightarrow$

$$\dot{k}(t) = e(t)^2 \text{ makes closed loop asymptotically stable}$$

*and $k(\cdot)$ remains bounded*

Problem: Disturbances or $y_{\text{ref}} \neq 0$ lead to unbounded $k(\cdot)$!

# High-gain-Feedback: linear case




Solution: Aim for practical stability, i.e. $|e(t)| \leq \lambda$ for $t >> 0$ and some small $\lambda > 0$

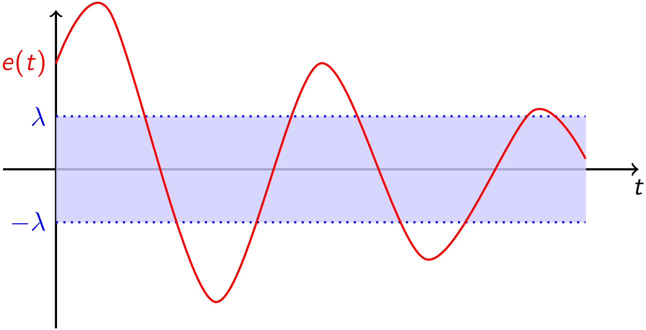
**Theorem ($\lambda$-tracking, Ilchmann & Ryan 1994)**

*Assume $cb > 0$, stable zero-dynamics and $y_{\text{ref}}, \dot{y}_{\text{ref}}$ bounded. For $\lambda > 0$ consider*

$$\dot{k}(t) = \begin{cases} |e(t)|\big(|e(t)| - \lambda\big), & |e(t)| > \lambda, \\ 0, & |e(t)| \leq \lambda. \end{cases}$$

*Then the closed loop is practically stable.*

# Remaining problems of $\lambda$-tracker




Problems:

- No guarantees when $|e(t)| \leq \lambda$
- No bounds on transient behaviour
- Monotonically growing $k(\cdot)$ $\Rightarrow$ Measurement noise unnecessarily amplified

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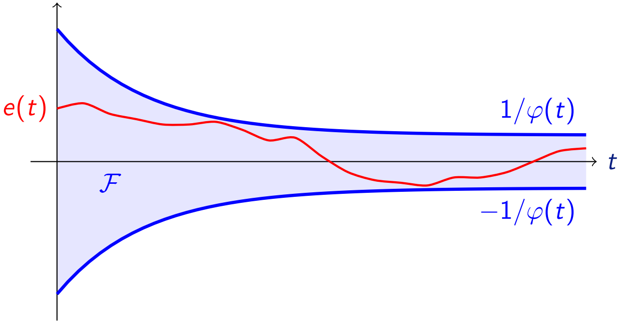
# The funnel as time-varying error bound

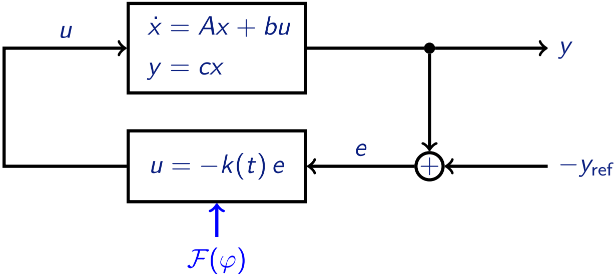
$$\mathcal{F} = \mathcal{F}(\varphi) := \{\ (t, e)\ \mid \varphi(t)|e| < 1\ \}$$




Idea: $k(t)$ large $\Leftrightarrow$ Distance of $e(t)$ to funnel boundary small

$$k(t) = \frac{1}{1 - \varphi(t)|e(t)|}$$

# Der lineare SISO Fall



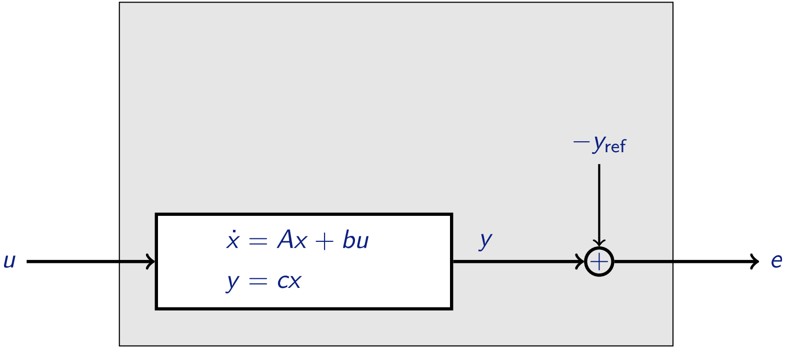

**Theorem (Funnel Control, Ilchmann, Ryan, Sangwin 2002)**

*Let* $cb > 0$, $A_{22}$ *Hurwitz,* $y_{\text{ref}}, \dot{y}_{\text{ref}}, \varphi, \dot{\varphi}$ *bounded,* $\liminf_{t\to\infty} \varphi(t) > 0$ *and* $e(0)\varphi(0) < 1$. *Then*

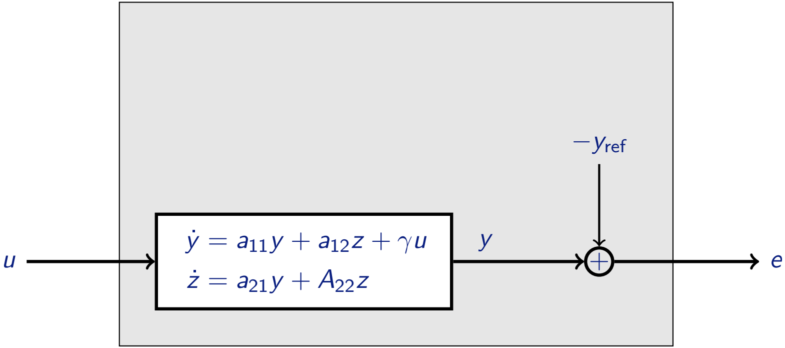
$$k(t) = \frac{1}{1 - \varphi(t)|e(t)|}$$

*remains bounded in the closed loop, i.e.* $e(t)$ *remains within funnel.*

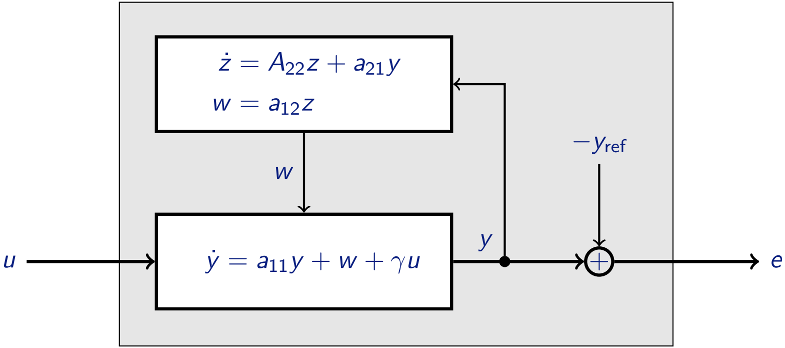
# Proof idea: Reinterpretation of system



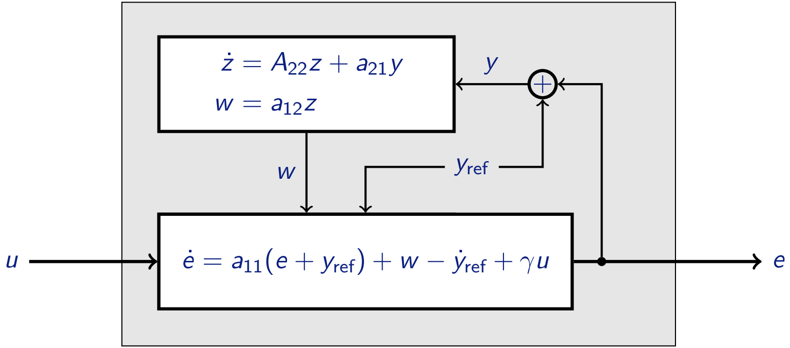
# Proof idea: Reinterpretation of system



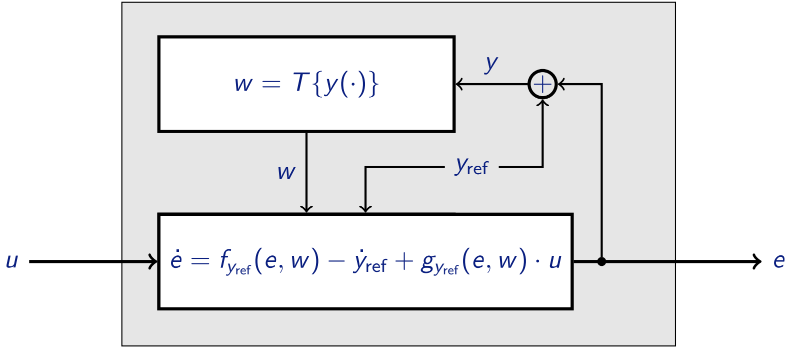
# Proof idea: Reinterpretation of system



# Proof idea: Reinterpretation of system



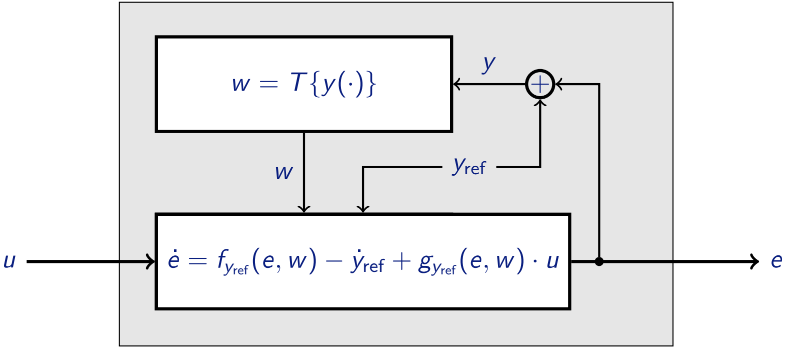
# Proof idea: Reinterpretation of system




Assumptions:

- $T$ is causal BIBO operator, i.e. $\exists \kappa(\cdot): \quad \|w\| \leq \kappa(\|y\|)$
- $f_{y_{\text{ref}}}$ and $g_{y_{\text{ref}}}$ continuous and $g_{y_{\text{ref}}} > 0$
- $y_{\text{ref}}$ and $\dot y_{\text{ref}}$ bounded

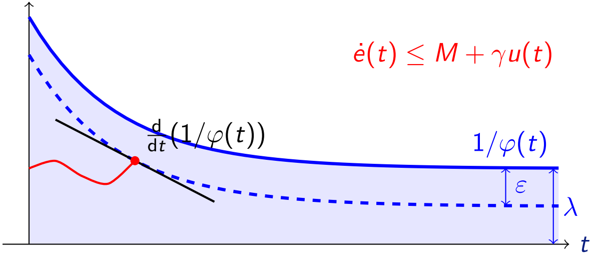
# Proof idea: Reinterpretation of system




Proof idea: Consider maximal solution $e : [0,\omega) \to \mathbb{R}$, it remains to show $\omega = \infty$.

- $e(t)$ within funnel $\Rightarrow$ $e$ bounded $\Rightarrow$ $y$ bounded $\Rightarrow$ $w$ bounded
- $\Rightarrow$ $f_{y_{\text{ref}}}(e,w)$ bounded, $g_{y_{\text{ref}}}(e,w)$ bounded away from zero
- Hence $\dot e(t) \leq M + \gamma u(t)$ if $u(t) < 0$ and $\dot e(t) \geq -M + \gamma u(t)$ if $u(t) > 0$
- In particular, $u(t) << 0 \Rightarrow \dot e(t) << 0$ and $u(t) >> 0 \Rightarrow \dot e(t) >> 0$

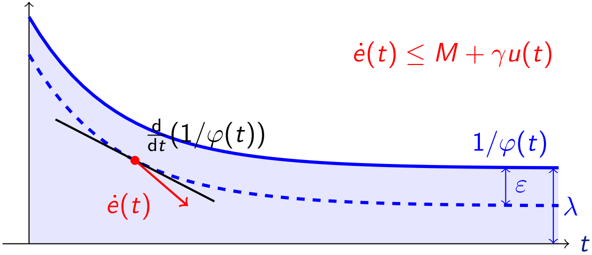
# Proof idea: Funnel invariant




Assumptions: $\varepsilon < \lambda/2$, $1/\varphi(t) \geq \lambda$, $\frac{\mathrm{d}}{\mathrm{d}t}(1/\varphi(t)) \geq -\Psi$

- $e(t) = \frac{1}{\varphi(t)} - \varepsilon \Rightarrow k(t) = \frac{1}{1-\varphi(t)|e(t)|} = \frac{1}{\varphi(t)\varepsilon} \geq \frac{\lambda}{\varepsilon}$
- $\Rightarrow u(t) = -k(t)e(t) \leq -\frac{\lambda^2}{2\varepsilon}$
- $\Rightarrow \dot e(t) \leq M + \gamma u(t) \leq M - \frac{\gamma\lambda^2}{2\varepsilon}$
- Hence $\varepsilon \leq \dfrac{\gamma\lambda^2}{2(\Psi + M)}$ implies that $\dot e(t) \leq -\Psi \leq \frac{\mathrm{d}}{\mathrm{d}t}(1/\varphi(t))$

# Proof idea: Funnel invariant




Consequence: For sufficiently small $\varepsilon > 0$,

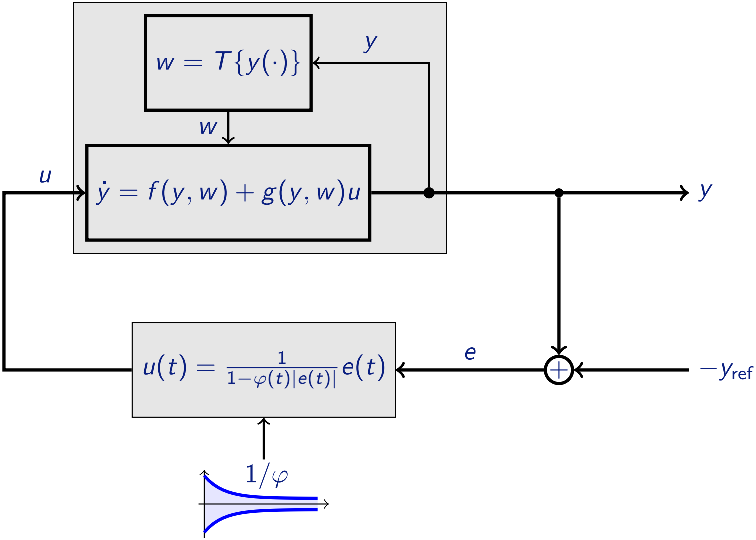
$$\mathcal{F}_\varepsilon := \{ \ (t, e) \ \mid |e(t)| < 1/\varphi(t) - \varepsilon \ \}$$

is positively invariant, hence

$$(0, e(0)) \in \mathcal{F}_\varepsilon \quad \Rightarrow \quad (t, e(t)) \in \mathcal{F}_\varepsilon \ \forall t \geq 0$$

and finite escape time is not possible!

# Summary of original Funnel Controller



# Contents

# Input saturations

$u$ physically bounded: $u(t) \in [-U, U]$ for $U > 0$

## Analysis of proof gives

$$U \geq \frac{\Psi + M}{\gamma} \quad \Rightarrow \quad \text{Funnel Controller works}$$

where

- $\frac{\mathrm{d}}{\mathrm{d}t}(1/\varphi(t)) \geq -\Psi$
- $|f(y, w) - \dot{y}_{\text{ref}}| \leq M$
- $g(y, w) \geq \gamma > 0$

## Reminder

$y_{\text{ref}}$ bounded $\wedge$ $e$ within funnel $\Rightarrow$ $y$ bounded $\Rightarrow$ $w$ bounded

cf.: Ilchmann & Trenn 2004; Hopfe, Ilchmann, Ryan 2010

# MIMO

**System class: MIMO systems**

System with $m$ inputs and $m$ outputs:

$$\dot{y} = f(y, w, u), \quad w = T\{y\}$$

with $T$ a causal BIBO-operator and

$$\frac{\langle u, f(y, w, u)\rangle}{\|u\|} \to \infty \quad \text{for } \|u\| \to \infty.$$

**Theorem (Ilchmann, Ryan, Sangwin 2002)**

*Funnel Controller*

$$u(t) = \frac{1}{1 - \varphi(t)\|e(t)\|} e(t)$$

*works, i.e.* $\|e(t)\| < 1/\varphi(t)$.

Proof idea: Instead of $\dot{e}$ consider $\frac{\mathrm{d}}{\mathrm{d}t}(\|e(t)\|^2)$

# Higher relative degree

**System class: Systems with well defined relative degree $r > 1$**

$$\begin{aligned} \dot{x}_1 &= x_2 + f_1(w, y) \\ \dot{x}_2 &= x_3 + f_2(w, y) \\ &\vdots \\ \dot{x}_r &= h(x_1, x_2, \ldots, x_r) + f_r(w, y) + \gamma u \end{aligned} \qquad \begin{aligned} w &= T\{y\} \\ y &= x_1 \end{aligned}$$

**Theorem (Ilchmann, Ryan, Townsend 2006, 2007)**

*Funnel Control*

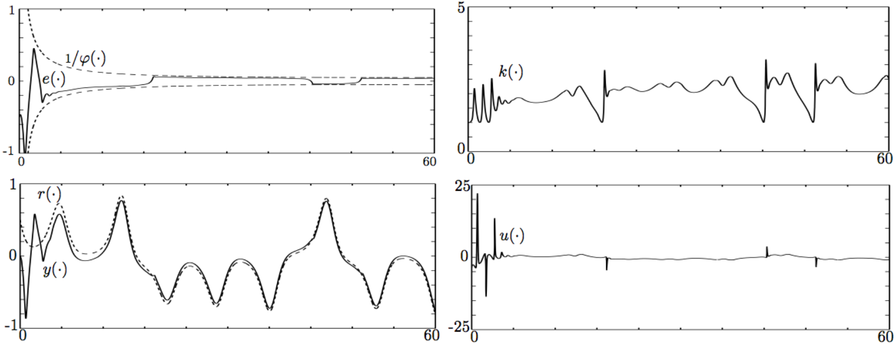
$$u(t) = -\gamma_r(k(t), e(t), \xi(t)), \qquad k(t) = \frac{1}{1 - \varphi(t)^2 \|e(t)\|^2}$$

*with suitable $\gamma_r(\cdot)$ and $\xi(\cdot)$ (Backstepping) works.*

# Example for relative degree two

For relative degree two systems the Funnel Controller is given by (simplified):

$$
\begin{aligned}
u(t) &= -k(t)e(t) - (\|e(t)\|^2 + k(t)^2)k(t)^4(1 + \|\xi(t)\|^2)(\xi(t) + k(t)e(t)) \\
k(t) &= 1/(1 - \varphi(t)^2\|e(t)\|^2) \\
\dot{\xi}(t) &= -\xi(t) + u(t)
\end{aligned}
$$




Taken from: Ilchmann, Ryan, Townsend 2007, SICON

# Alternative Approach for relative degree two

Use two funnels, one for error and one for derivative of error

## Simple Control Law

$$u(t) = -k_0(t)^2 e(t) - k_1(t)\dot{e}(t)$$

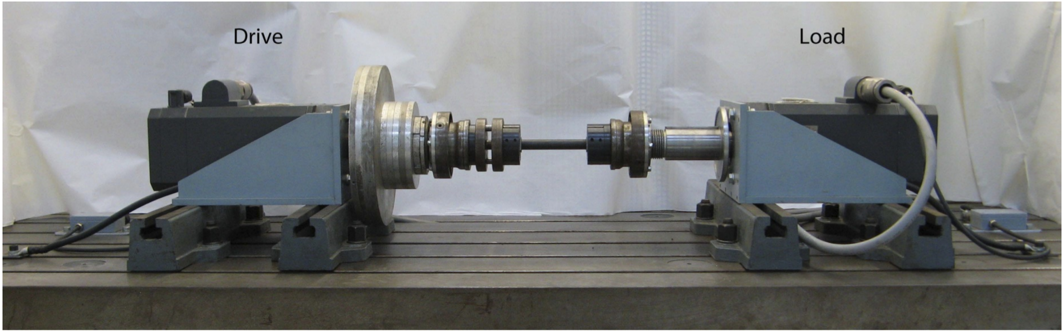
$$k_i(t) = \frac{1}{1/\varphi_i(t) - |e(t)|}, \quad i = 0, 1$$

System class: $\ddot{y}(t) = f(p_f(t), T_f\{y, \dot{y}\}(t)) + g(p_g(t), T_g\{y, \dot{y}\}(t))u(t)$

## Theorem (Hackl, Hopfe, Ilchmann, Mueller, Trenn 2012)

*The above Funnel Controller for relative-degree-two-systems works (under mild assumptions on $\varphi_0$ and $\varphi_1$).*

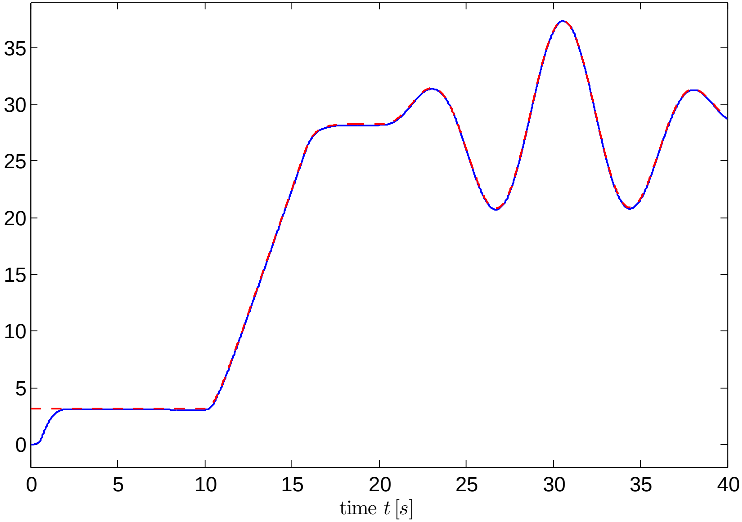
# Experimental verification




$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \gamma \end{bmatrix} \big(u(t) + u_L(t) - (Tx_2)(t)\big),$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t),$$

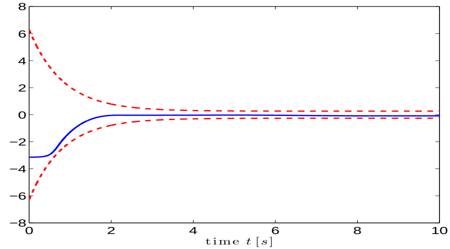
$x_1$: angle of rotating machine
$x_2 = \dot{x}_1$: angular velocity
$u_L$: unknown load
$T : \mathcal{C}(\mathbb{R}_{\geq 0} \to \mathbb{R}) \to \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ friction operator

# Tracking control in experiment

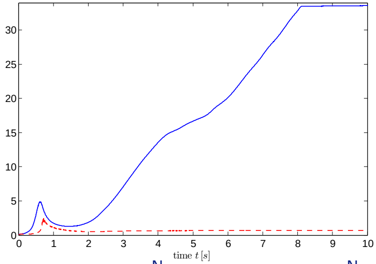



Measured angle $y(t)$ in rad, reference angle $y_{\mathrm{ref}}(t)$ in rad

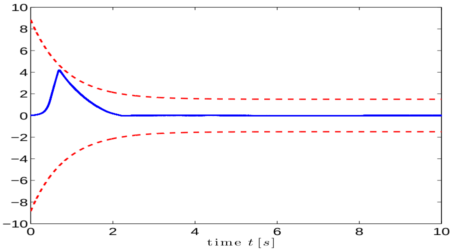
# Experiment: Error, gains, input



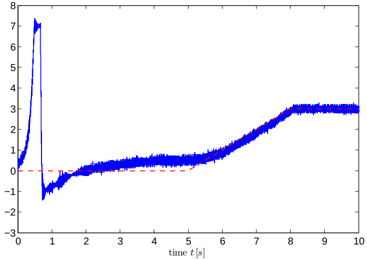

$e(t)$ in rad, $1/\varphi_0(t)$




$k_0(t)$ in $\frac{\text{Nm}}{\text{rad}}$, $k_1(t)$ in $\frac{\text{Nms}}{\text{rad}}$

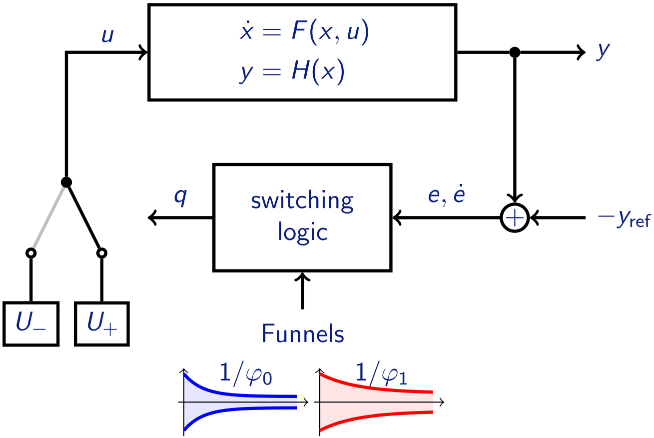



$\dot{e}(t)$ in rad/s, $1/\varphi_1(t)$

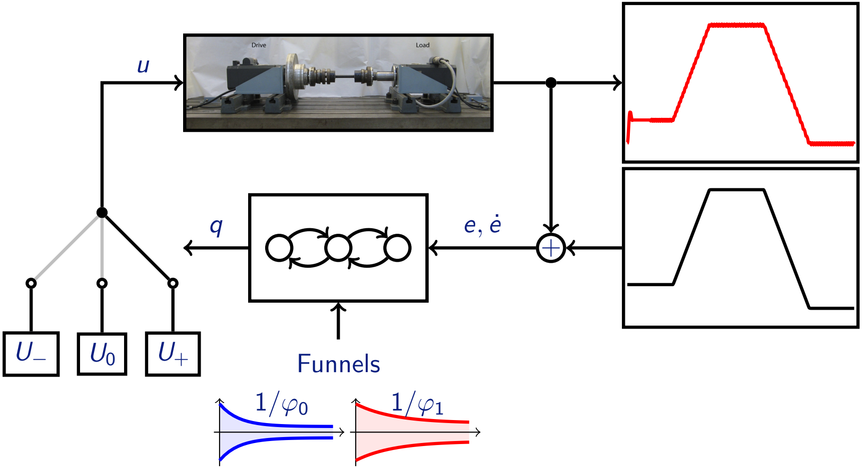



$u(t)$ in Nm, $u_L(t)$ in Nm

# Bang-Bang Funnel Control, Liberzon, Trenn 2010

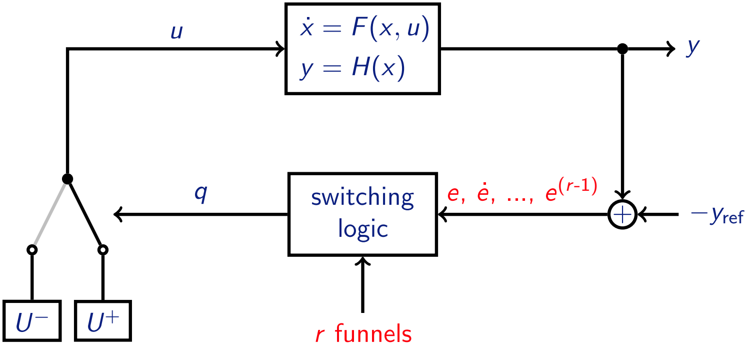


# Bang-Bang Funnel Control, Liberzon, Trenn 2010




This variant: Hackl, Trenn 2012

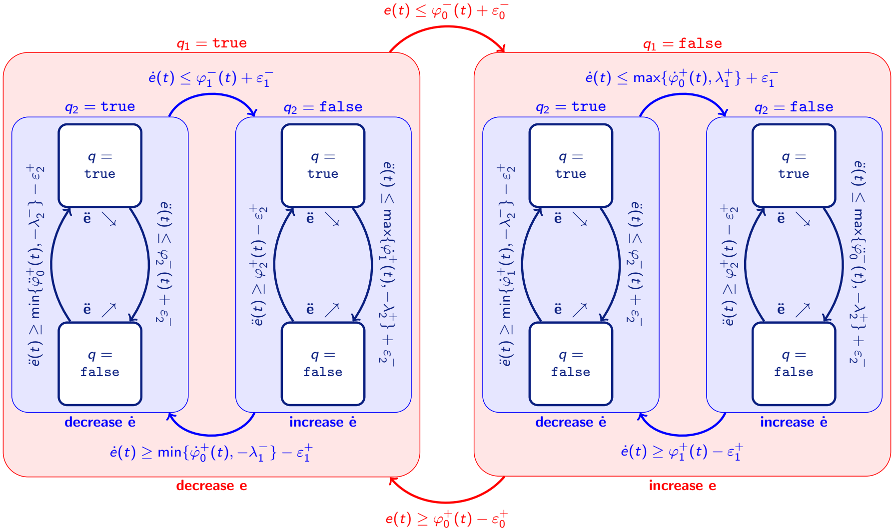
# Bang-Bang-Funnel-Controller for arbitrary high relative degree



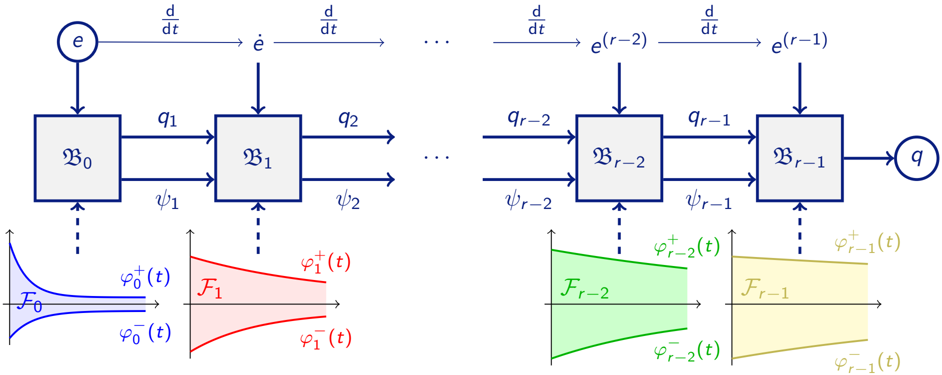

Liberzon & Trenn 2013, IEEE TAC

Skip details

# Explicit switching logic for $r = 3$



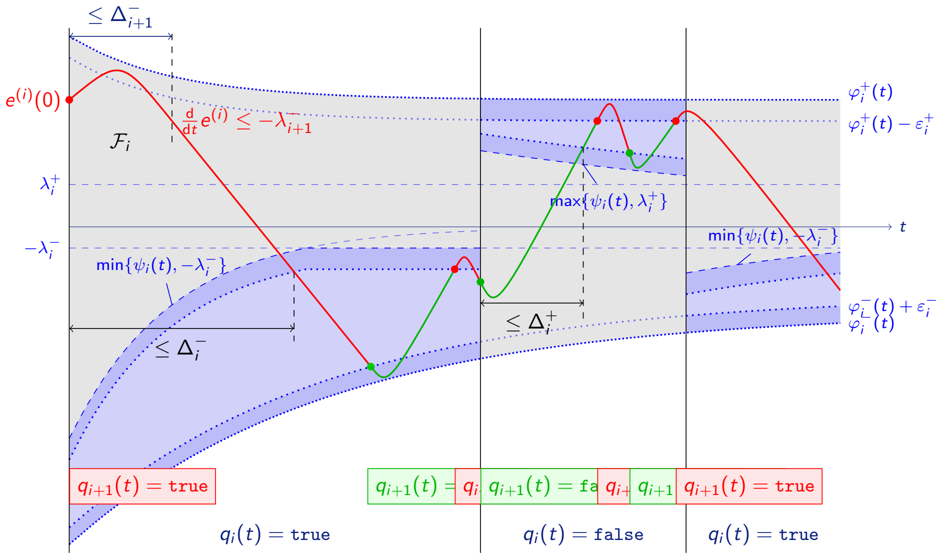
# Recursive switching logic



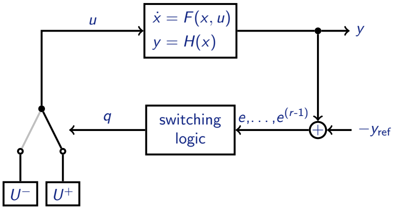

$q_i = \texttt{true} \quad \Rightarrow \quad$ Goal: $e^{(i)}(t) < \min\{\psi_i(t), -\lambda_i^-\}$

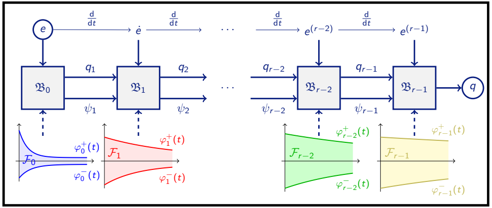
$q_i = \texttt{false} \quad \Rightarrow \quad$ Goal: $e^{(i)}(t) > \max\{\psi_i(t), \lambda_i^+\}$

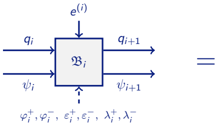
# Illustration of switching logic of block $\mathfrak{B}_i$

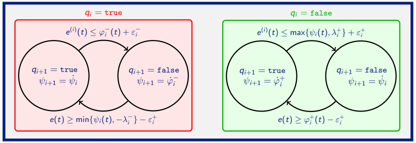


# Bang-Bang-Funnel-Controller main result




**Theorem (Liberzon & Trenn 2013)**

- *Feasibility (F1)-(F9)* $\Rightarrow$ *bang-bang funnel controller works*
- *Almost arbitrary $\mathcal{F}_0$ + BIBO zero dynamics + boundedness of $y_{\mathrm{ref}}$*
  $\Rightarrow$ *Feasibility holds with sufficiently large $U^+$ and $U^-$*

Skip simulation

# Simulation for $r = 4$

Example (academic), finite escape time for $y$ possible:

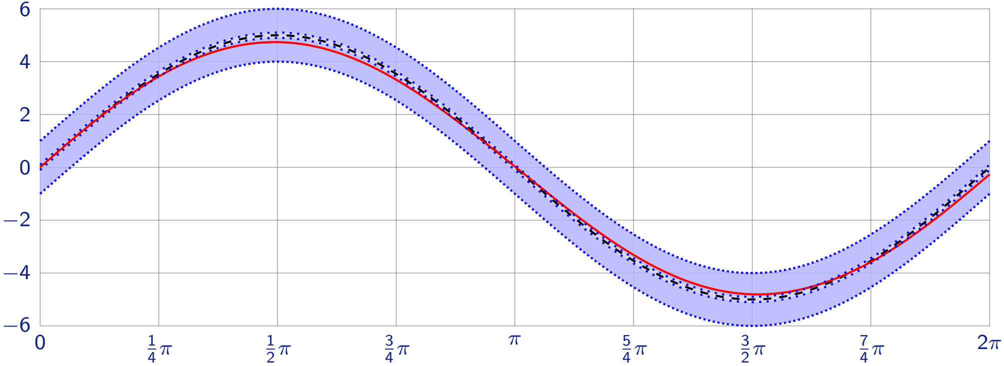
$$\begin{aligned} y^{(4)} &= z\,\dddot{y}^2 + e^z u, & y^{(i)}(0) &= y_{\text{ref}}^{(i)}(0),\ i = 0,1,2,3, \\ \dot{z} &= z(a-z)(z+b) - cy, & z(0) &= 0, \\ y_{\text{ref}}(t) &= 5\sin(t) \end{aligned}$$

controller parameter (constant funnels):

$$\begin{aligned}
&\varphi_0^+ = -\varphi_0^- \equiv 1, && \varepsilon_0^+ = \varepsilon_0^- = 0.9, && && \Delta_0^+ = \Delta_0^- = \infty, \\
&\varphi_1^+ = -\varphi_1^- \equiv 0.5, && \varepsilon_1^+ = \varepsilon_1^- = 0.1, && \lambda_1^+ = \lambda_1^- = 0, && \Delta_1^+ = \Delta_1^- = \Delta_0^\pm/2 = \infty, \\
&\varphi_2^+ = -\varphi_2^- \equiv 0.5, && \varepsilon_2^+ = \varepsilon_2^- = 0.1, && \lambda_2^+ = \lambda_2^- = 0.2, && \Delta_2^+ = \Delta_2^- = 0.4, \\
&\varphi_3^+ = -\varphi_3^- \equiv 4.5, && \varepsilon_3^+ = \varepsilon_3^- = 0.1, && \lambda_3^+ = \lambda_3^- = 4, && \Delta_3^+ = \Delta_3^- = 0.1, \\
& && && \lambda_4^+ = \lambda_4^- = 102, && \Delta_4^+ = \Delta_4^- = 0.0001.
\end{aligned}$$

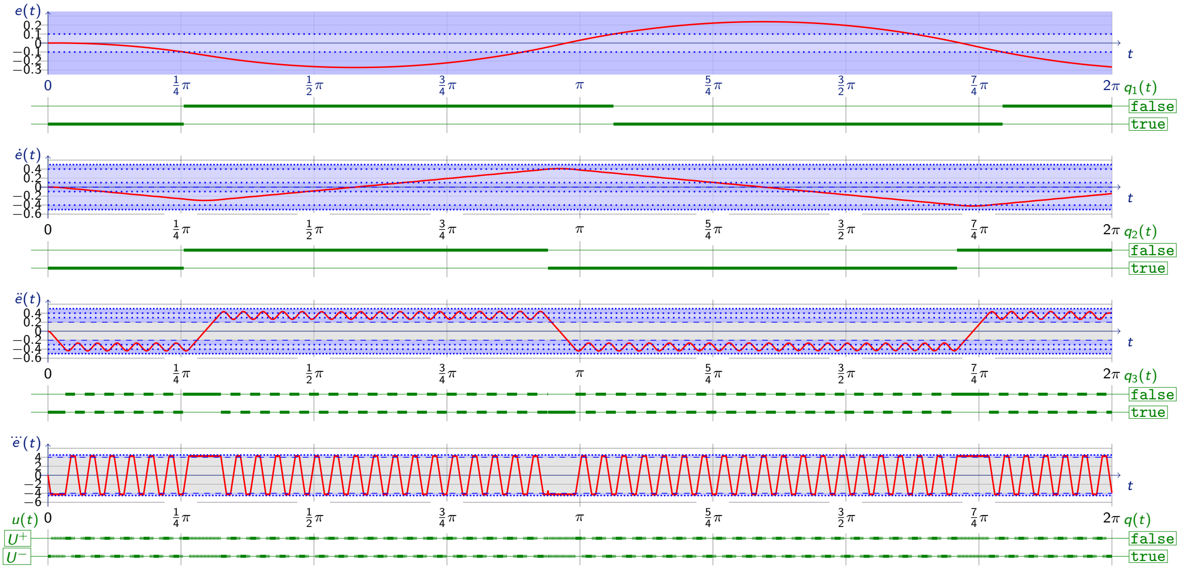
$$U^+ = -U^- = 254$$

# Simulation results, reference tracking




Switching frequency: up to 1000 Hz
Total number of switchings: ca. 2200

# Simulation results, error plots



# Historical Summary

- before 2002: High-gain-adaptive control
  - $\dot{k} = \|y\|^2$ (Morse 83; Byrnes & Willems 84; Mareels 84)
  - $\lambda$-tracking, (Ilchmann, Ryan 94, further results until 2002)
- 1991: The Miller-Davison-"Funnel"
  - only two error bounds: overshoot bound and final accuracy
  - $k(t)$ piecewise constant, monotonically increasing
  - frequency domain analysis
  - works for arbitrary relative degree
- since 2002: The Funnel Controller
  - Relative degree 1, nonlinear, MIMO (Ilchmann, Ryan, Sangwin 2002)
  - Input Constraints, Relativgrad 1, SISO+MIMO (Ilchmann, Trenn 2004; Hopfe, Ilchmann, Ryan 2010)
  - Higher relative degree, MIMO, via Backstepping (Ilchmann, Ryan, Townsend 2006, 2007)
  - Input constraints, Relativgrad 2, SISO (Liberzon, Trenn 2010 (Bang-Bang); Hackl, Hopfe, Ilchmann, Mueller, Trenn 2012)
  - Bang-Bang, arbitrary relative degree, SISO (Liberzon, Trenn 2013)