

Switched differential algebraic equations: Jumps and impulses

Stephan Trenn

Technomathematics group, University of Kaiserslautern, Germany

Lothar-Collatz-Kolloquium für Angewandte Mathematik
Universität Hamburg, 06.11.2014



Contents

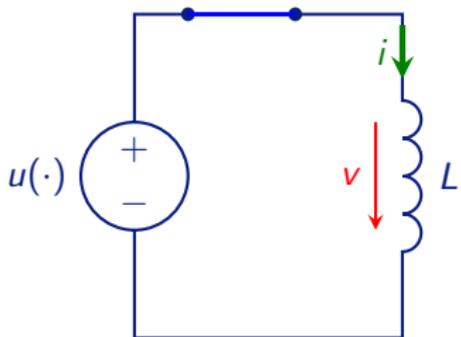


- 1 Introduction
- 2 Switched DAEs: Solution Theory
 - Definition
 - Review: classical distribution theory
 - Restriction of distributions
 - Piecewise smooth distributions
 - Distributional solutions
 - Impulse-freeness
- 3 Observability
 - Definition
 - The single switch result
 - Calculation of the four subspaces
 - \mathcal{C}_-
 - O_- and O_+^-
 - O_+^{imp}
- 4 Summary

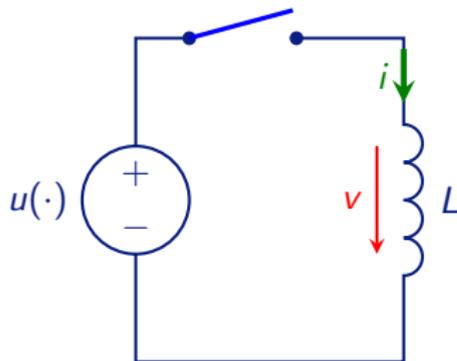


Motivating example

$t < 0$



$t \geq 0$



inductivity law:

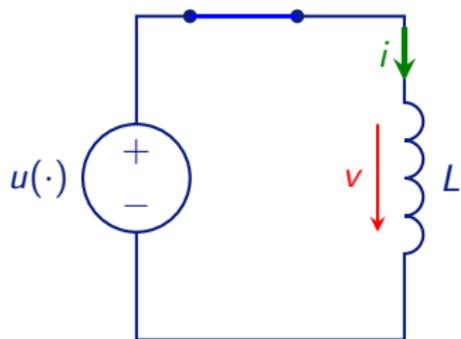
$$L \frac{d}{dt} i = v$$

switch dependent: $0 = v - u$

$$0 = i$$

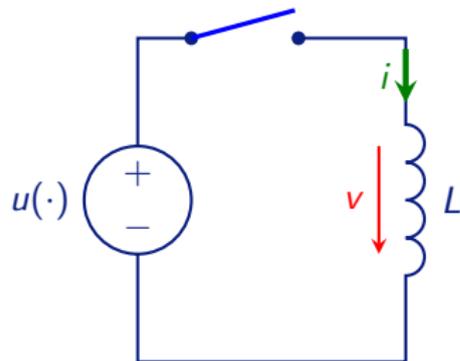


Motivating example

 $t < 0$


$$x = [i, v]^T$$

$$\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u$$

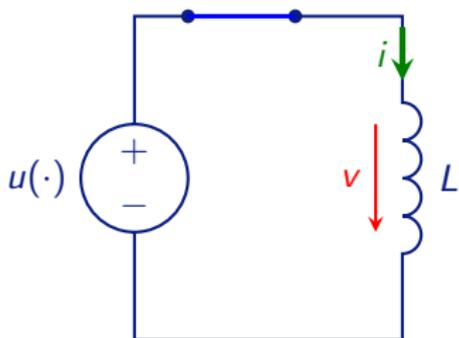
 $t \geq 0$


$$x = [i, v]^T$$

$$\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

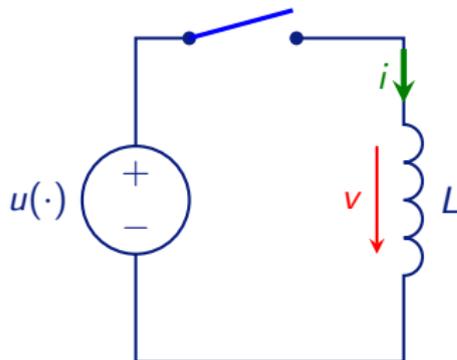


Motivating example

 $t < 0$


$$E_1 \dot{x} = A_1 x + B_1 u$$

on $(-\infty, 0)$

 $t \geq 0$


$$E_2 \dot{x} = A_2 x + B_2 u$$

on $[0, \infty)$

→ switched differential-algebraic equation

Solution of circuit example



$$t < 0$$

$$v = u$$

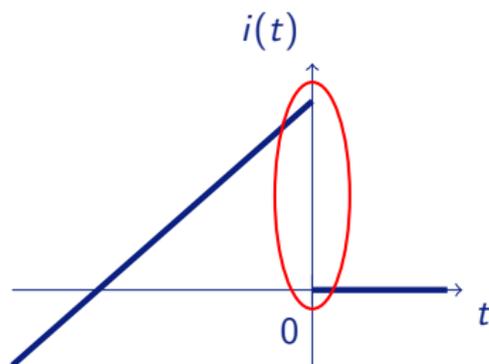
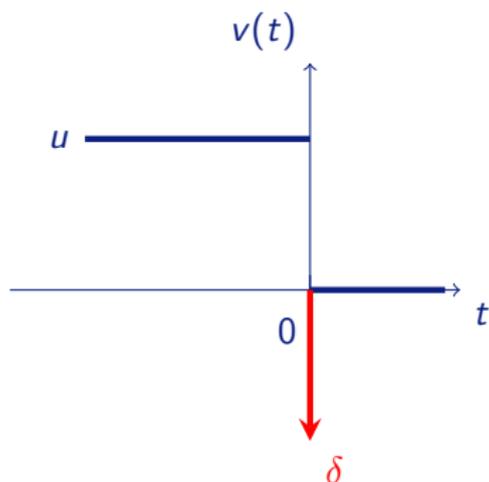
$$L \frac{d}{dt} i = v$$

$$t \geq 0$$

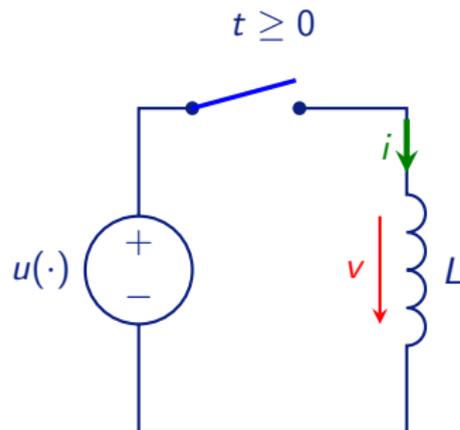
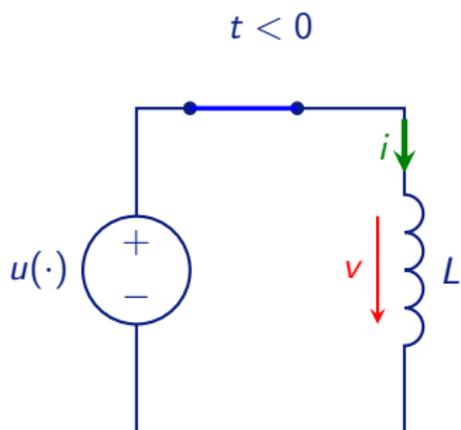
$$i = 0$$

$$v = L \frac{d}{dt} i$$

Solution (assume constant input u):



Observations



Observations

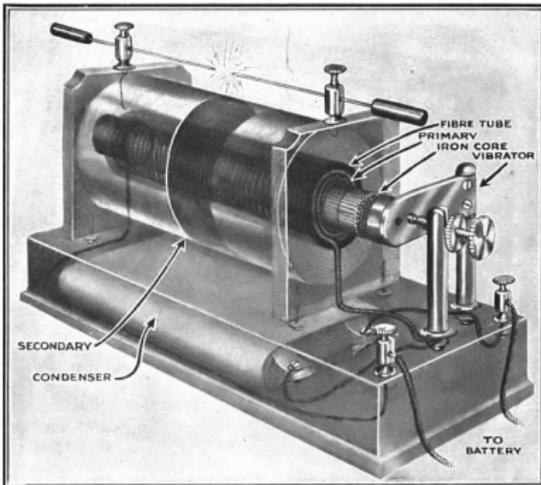
- $x(0^-) \neq 0$ inconsistent for $E_2 \dot{x} = A_2 x + B_2 u$
- unique jump from $x(0^-)$ to $x(0^+)$
- derivative of jump = Dirac impulse appears in solution

Dirac impulse is “real”



Dirac impulse

Not just a mathematical artifact!



Drawing: Harry Winfield Secor, public domain

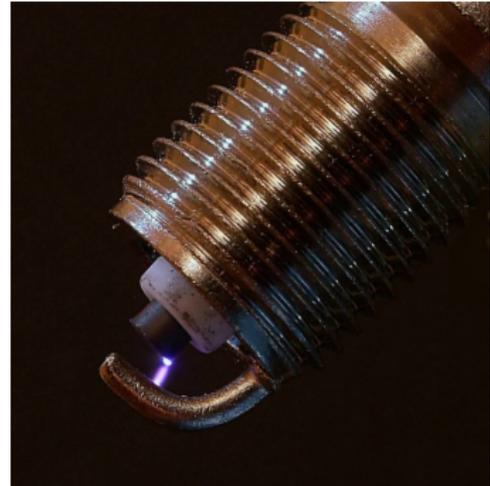


Foto: Ralf Schumacher, CC-BY-SA 3.0

Contents



- 1 Introduction
- 2 Switched DAEs: Solution Theory
 - Definition
 - Review: classical distribution theory
 - Restriction of distributions
 - Piecewise smooth distributions
 - Distributional solutions
 - Impulse-freeness
- 3 Observability
 - Definition
 - The single switch result
 - Calculation of the four subspaces
 - \mathcal{C}_-
 - O_- and O_+^-
 - O_+^{imp}
- 4 Summary

Definition



Switch \rightarrow Different DAE models (=modes)
depending on **time-varying** position of switch

Definition (Switched DAE)

Switching signal $\sigma : \mathbb{R} \rightarrow \{1, \dots, N\}$ picks mode at each time $t \in \mathbb{R}$:

$$\begin{aligned} E_{\sigma(t)} \dot{x}(t) &= A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t) \\ y(t) &= C_{\sigma(t)} x(t) + D_{\sigma(t)} u(t) \end{aligned} \quad (\text{swDAE})$$

Attention

Each mode might have **different consistency spaces**
 \Rightarrow inconsistent initial values at each switch
 \Rightarrow Dirac impulses, in particular **distributional solutions**

Definition



Switch \rightarrow Different DAE models (=modes)
depending on **time-varying** position of switch

Definition (Switched DAE)

Switching signal $\sigma : \mathbb{R} \rightarrow \{1, \dots, N\}$ picks mode at each time $t \in \mathbb{R}$:

$$\begin{aligned} E_{\sigma} \dot{x} &= A_{\sigma} x + B_{\sigma} u \\ y &= C_{\sigma} x + D_{\sigma} u \end{aligned} \quad (\text{swDAE})$$

Attention

Each mode might have **different consistency spaces**
 \Rightarrow inconsistent initial values at each switch
 \Rightarrow Dirac impulses, in particular **distributional solutions**

Distribution theory - basic ideas



Distributions - overview

- Generalized functions
- Arbitrarily often differentiable
- Dirac-Impulse δ is “derivative” of Heaviside step function $\mathbb{1}_{[0,\infty)}$

Two different formal approaches

- 1 Functional analytical: Dual space of the space of test functions (L. Schwartz 1950)
- 2 Axiomatic: Space of all “derivatives” of continuous functions (J. Sebastião e Silva 1954)

Distributions - formal



Definition (Test functions)

$$\mathcal{C}_0^\infty := \{ \varphi : \mathbb{R} \rightarrow \mathbb{R} \mid \varphi \text{ is smooth with compact support} \}$$

Definition (Distributions)

$$\mathbb{D} := \{ D : \mathcal{C}_0^\infty \rightarrow \mathbb{R} \mid D \text{ is linear and continuous} \}$$

Definition (Regular distributions)

$$f \in L_{1,\text{loc}}(\mathbb{R} \rightarrow \mathbb{R}): \quad f_{\mathbb{D}} : \mathcal{C}_0^\infty \rightarrow \mathbb{R}, \quad \varphi \mapsto \int_{\mathbb{R}} f(t)\varphi(t)dt \in \mathbb{D}$$

Definition (Derivative)

$$D'(\varphi) := -D(\varphi')$$

Dirac Impulse at $t_0 \in \mathbb{R}$

$$\delta_{t_0} : \mathcal{C}_0^\infty \rightarrow \mathbb{R}, \quad \varphi \mapsto \varphi(t_0)$$

Multiplication with functions



Definition (Multiplication with smooth functions)

$$\alpha \in \mathcal{C}^\infty : (\alpha D)(\varphi) := D(\alpha\varphi)$$

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned} \quad (\text{swDAE})$$

Coefficients not smooth

Problem: $E_\sigma, A_\sigma, C_\sigma \notin \mathcal{C}^\infty$

Observation, for $\sigma_{[t_i, t_{i+1})} \equiv p_i, i \in \mathbb{Z}$:

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned} \Leftrightarrow \forall i \in \mathbb{Z} : \begin{aligned} (E_{p_i} \dot{x})_{[t_i, t_{i+1})} &= (A_{p_i} x + B_{p_i} u)_{[t_i, t_{i+1})} \\ y_{[t_i, t_{i+1})} &= (C_{p_i} x + D_{p_i} u)_{[t_i, t_{i+1})} \end{aligned}$$

New question: **Restriction of distributions**

Desired properties of distributional restriction



Distributional restriction:

$$\{ M \subseteq \mathbb{R} \mid M \text{ interval} \} \times \mathbb{D} \rightarrow \mathbb{D}, \quad (M, D) \mapsto D_M$$

and for each interval $M \subseteq \mathbb{R}$

- 1 $D \mapsto D_M$ is a **projection** (linear and idempotent)
- 2 $\forall f \in L_{1,loc} : (f_{\mathbb{D}})_M = (f_M)_{\mathbb{D}}$
- 3 $\forall \varphi \in C_0^\infty : \left[\begin{array}{ll} \text{supp } \varphi \subseteq M & \Rightarrow D_M(\varphi) = D(\varphi) \\ \text{supp } \varphi \cap M = \emptyset & \Rightarrow D_M(\varphi) = 0 \end{array} \right]$
- 4 $(M_i)_{i \in \mathbb{N}}$ pairwise disjoint, $M = \bigcup_{i \in \mathbb{N}} M_i$:

$$D_M = \sum_{i \in \mathbb{N}} D_{M_i}, \quad D_{M_1 \dot{\cup} M_2} = D_{M_1} + D_{M_2}, \quad (D_{M_1})_{M_2} = 0$$

Theorem ([T. 2009])

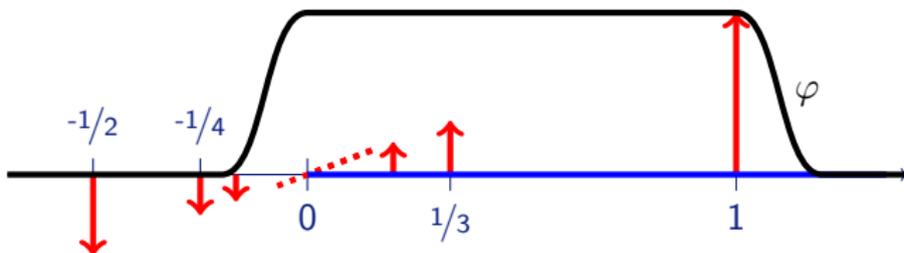
*Such a distributional restriction **does not exist**.*

Proof of non-existence of restriction



Consider the following (well defined!) distribution:

$$D := \sum_{i \in \mathbb{N}} d_i \delta_{d_i}, \quad d_i := \frac{(-1)^i}{i+1}$$



Restriction should give

$$D_{[0, \infty)} = \sum_{k \in \mathbb{N}} d_{2k} \delta_{d_{2k}}$$

Choose $\varphi \in \mathcal{C}_0^\infty$ such that $\varphi_{[0,1]} \equiv 1$:

$$D_{[0, \infty)}(\varphi) = \sum_{k \in \mathbb{N}} d_{2k} = \sum_{k \in \mathbb{N}} \frac{1}{2k+1} = \infty$$

Dilemma



Switched DAEs

- Examples: distributional solutions
- Multiplication with non-smooth coefficients
- Or: Restriction on intervals

Distributions

- Distributional restriction not possible
- Multiplication with non-smooth coefficients not possible
- *Initial value problems cannot be formulated*

Underlying problem

Space of distributions **too big**.

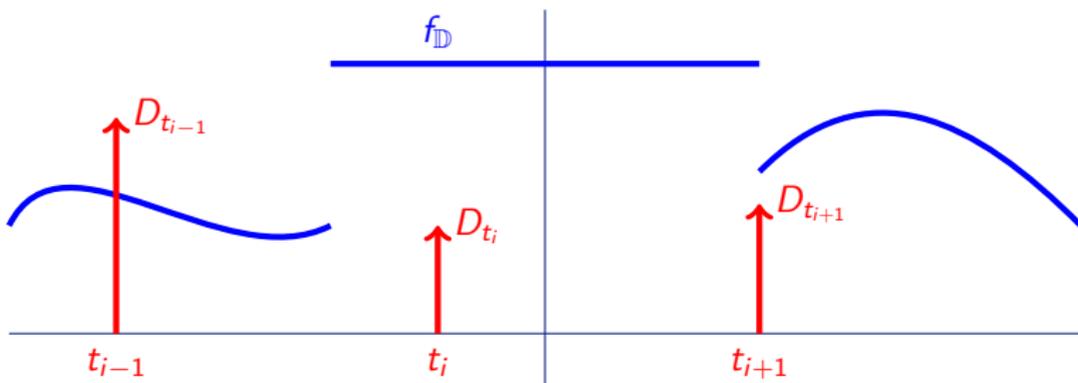
Piecewise smooth distributions



Define a suitable smaller space:

Definition (Piecewise smooth distributions \mathbb{D}_{pwC^∞} , [T. 2009])

$$\mathbb{D}_{pwC^\infty} := \left\{ f_{\mathbb{D}} + \sum_{t \in T} D_t \mid \begin{array}{l} f \in C_{pw}^\infty, \\ T \subseteq \mathbb{R} \text{ locally finite,} \\ \forall t \in T : D_t = \sum_{i=0}^{n_t} a_i^t \delta_t^{(i)} \end{array} \right\}$$



Properties of $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$



- $\mathcal{C}_{\text{pw}}^\infty$ “ \subseteq ” $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$
- $D \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty} \Rightarrow D' \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$
- **Well defined restriction** $\mathbb{D}_{\text{pw}\mathcal{C}^\infty} \rightarrow \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$

$$D = f_{\mathbb{D}} + \sum_{t \in T} D_t \quad \mapsto \quad D_M := (f_M)_{\mathbb{D}} + \sum_{t \in T \cap M} D_t$$

- **Multiplication** with $\alpha = \sum_{i \in \mathbb{Z}} \alpha_i [t_i, t_{i+1}) \in \mathcal{C}_{\text{pw}}^\infty$ well defined:

$$\alpha D := \sum_{i \in \mathbb{Z}} \alpha_i D_{[t_i, t_{i+1})}$$

- **Evaluation** at $t \in \mathbb{R}$: $D(t^-) := f(t^-)$, $D(t^+) := f(t^+)$
- **Impulses** at $t \in \mathbb{R}$: $D[t] := \begin{cases} D_t, & t \in T \\ 0, & t \notin T \end{cases}$

Application to (swDAE)

(x, u) solves (swDAE) \Leftrightarrow (swDAE) holds in $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$

Relevant questions



$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned} \quad (\text{swDAE})$$

Piecewise-smooth distributional solution framework

$$x \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}^n, \quad u \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}^m, \quad y \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}^p$$

- Existence and uniqueness of solutions?
- Jumps and impulses in solutions?
- Conditions for impulse free solutions?
- Control theoretical questions
 - Stability and stabilization
 - Observability and observer design
 - Controllability and controller design

Existence and uniqueness of solutions for (swDAE)



$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u \quad (\text{swDAE})$$

Basic assumptions

- $\sigma \in \Sigma_0 := \left\{ \sigma : \mathbb{R} \rightarrow \{1, \dots, N\} \mid \begin{array}{l} \sigma \text{ is piecewise constant and} \\ \sigma|_{(-\infty, 0)} \text{ is constant} \end{array} \right\}$.
- (E_p, A_p) is **regular** $\forall p \in \{1, \dots, N\}$, i.e. $\det(sE_p - A_p) \neq 0$

Theorem (T. 2009)

Consider (swDAE) with regular (E_p, A_p) . Then

$$\forall u \in \mathbb{D}_{\text{pwc}}^m \quad \forall \sigma \in \Sigma_0 \quad \exists \text{ solution } x \in \mathbb{D}_{\text{pwc}}^n$$

and $x(0^-)$ **uniquely** determines x .

Inconsistent initial values



$$E\dot{x} = Ax + Bu, \quad x(0) = x^0 \in \mathbb{R}^n$$

Inconsistent initial value = special switched DAE

$$\begin{aligned} \dot{x}_{(-\infty,0)} &= 0, & x(0^-) &= x^0 \\ (E\dot{x})_{[0,\infty)} &= (Ax + Bu)_{[0,\infty)} \end{aligned}$$

Corollary (Consistency projector)

Exist *unique* consistency projector $\Pi_{(E,A)}$ such that

$$x(0^+) = \Pi_{(E,A)}x^0$$

$\Pi_{(E,A)}$ can easily be calculated via the **Wong sequences** [T. 2009].

Sufficient conditions for impulse-freeness



Question

When are **all solutions** of homogenous (swDAE) $E_\sigma \dot{x} = A_\sigma x$ **impulse free**?

Note: Jumps are OK.

Lemma (Sufficient conditions)

- (E_p, A_p) all have **index one** (i.e. $(sE - A)^{-1}$ is proper)
 \Rightarrow (swDAE) impulse free
- all **consistency spaces** of (E_p, A_p) **coincide**
 \Rightarrow (swDAE) impulse free

Characterization of impulse-freeness



Theorem (Impulse-freeness, [T. 2009])

The switched DAE $E_\sigma \dot{x} = A_\sigma x$ is **impulse free** $\forall \sigma \in \Sigma_0$

$$\Leftrightarrow E_q(I - \Pi_q)\Pi_p = 0 \quad \forall p, q \in \{1, \dots, N\}$$

where $\Pi_p := \Pi_{(E_p, A_p)}$, $p \in \{1, \dots, N\}$ is the p -th consistency projector.

Remark

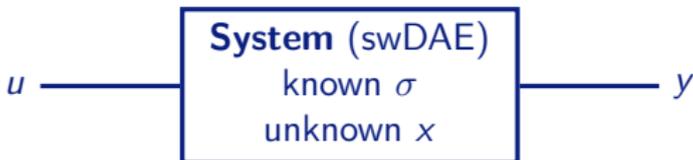
- Index-1-case $\Rightarrow E_q(I - \Pi_q) = 0 \quad \forall q$
- Consistency spaces equal $\Rightarrow (I - \Pi_q)\Pi_p = 0 \quad \forall p, q$

Contents



- 1 Introduction
- 2 Switched DAEs: Solution Theory
 - Definition
 - Review: classical distribution theory
 - Restriction of distributions
 - Piecewise smooth distributions
 - Distributional solutions
 - Impulse-freeness
- 3 Observability
 - Definition
 - The single switch result
 - Calculation of the four subspaces
 - \mathcal{E}_-
 - O_- and O_+^-
 - O_+^{imp}
- 4 Summary

Global Observability of Switched DAEs

**Definition (Global observability)**

(swDAE) with given σ is **(globally) observable** $:\Leftrightarrow$

\forall solutions $(u_1, x_1, y_1), (u_2, x_2, y_2)$: $(u_1, y_1) \equiv (u_2, y_2) \Rightarrow x_1 \equiv x_2$

Lemma (0-distinguishability)

(swDAE) is observable if, and only if,

$$y \equiv 0 \text{ and } u \equiv 0 \Rightarrow x \equiv 0.$$

Hence consider in the following (swDAE) without inputs:

$$\begin{array}{l} E_{\sigma} \dot{x} = A_{\sigma} x \\ y = C_{\sigma} x \end{array}$$

and observability question:

$$y \equiv 0 \stackrel{?}{\Rightarrow} x \equiv 0$$



Motivating example

System 1:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

$$y = x_3, \dot{y} = \dot{x}_3 = 0, x_2 = 0, \dot{x}_1 = 0$$

$$\Rightarrow x_1 \text{ unobservable}$$

System 2:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

$$y = x_3 = \dot{x}_1, x_1 = 0, \dot{x}_2 = 0$$

$$\Rightarrow x_2 \text{ unobservable}$$

$$\sigma(\cdot) : 1 \rightarrow 2$$

Jump in x_1 produces impulse in y
 \Rightarrow Observability

$$\sigma(\cdot) : 2 \rightarrow 1$$

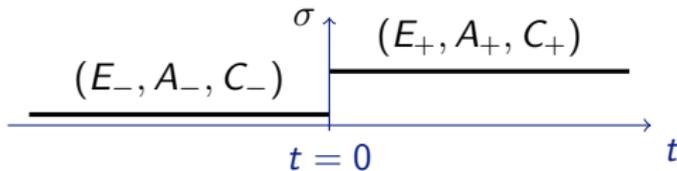
Jump in x_2 no influence in y
 \Rightarrow x_2 remains unobservable

Question

$$E_p \dot{x} = A_p x + B_p u \quad \text{not observable} \quad \stackrel{?}{\Rightarrow} \quad E_\sigma \dot{x} = A_\sigma x + B_\sigma u \quad \text{observable}$$

$$y = C_p x + D_p u \quad \text{observable} \quad \Rightarrow \quad y = C_\sigma x + D_\sigma u \quad \text{observable}$$

The single switch result



Theorem (Unobservable subspace, Tanwani & T. 2010)

For (swDAE) with a **single switch** the following equivalence holds

$$y \equiv 0 \Leftrightarrow x(0^-) \in \mathcal{M}$$

where

$$\mathcal{M} := \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\text{imp}}$$

In particular: (swDAE) observable $\Leftrightarrow \mathcal{M} = \{0\}$.

What are these four subspace?

The four subspaces



Unobservable subspace: $\mathcal{M} := \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\text{imp}}$, i.e.

$$x(0^-) \in \mathcal{M} \Leftrightarrow y_{(-\infty,0)} \equiv 0 \wedge y[0] = 0 \wedge y_{(0,\infty)} \equiv 0$$

The four spaces

- Consistency: $x(0^-) \in \mathfrak{C}_-$
- Left unobservability: $y_{(-\infty,0)} \equiv 0 \Leftrightarrow x(0^-) \in \ker O_-$
- Right unobservability: $y_{(0,\infty)} \equiv 0 \Leftrightarrow x(0^-) \in \ker O_+^-$
- Impulse unobservability: $y[0] = 0 \Leftrightarrow x(0^-) \in \ker O_+^{\text{imp}}$

Question

How to calculate these four spaces?

Wong sequences



Definition

Let $E, A \in \mathbb{R}^{m \times n}$. The corresponding Wong sequences of the pair (E, A) are:

$$\begin{aligned} \mathcal{V}_0 &:= \mathbb{R}^n, & \mathcal{V}_{i+1} &:= A^{-1}(E\mathcal{V}_i), & i &= 0, 1, 2, 3, \dots \\ \mathcal{W}_0 &:= \{0\}, & \mathcal{W}_{j+1} &:= E^{-1}A(\mathcal{W}_j), & j &= 0, 1, 2, 3, \dots \end{aligned}$$

Note: $M^{-1}\mathcal{S} := \{ x \mid Mx \in \mathcal{S} \}$ and $M\mathcal{S} := \{ Mx \mid x \in \mathcal{S} \}$

Clearly, $\exists i^*, j^* \in \mathbb{N}$

$$\mathcal{V}_0 \supset \mathcal{V}_1 \supset \dots \supset \mathcal{V}_{i^*} = \mathcal{V}_{i^*+1} = \mathcal{V}_{i^*+2} = \dots$$

$$\mathcal{W}_0 \subset \mathcal{W}_1 \subset \dots \subset \mathcal{W}_{j^*} = \mathcal{W}_{j^*+1} = \mathcal{W}_{j^*+2} = \dots$$

Wong limits:

$$\mathcal{V}^* := \bigcap_{i \in \mathbb{N}} \mathcal{V}_i = \mathcal{V}_{i^*}$$

$$\mathcal{W}^* = \bigcup_{j \in \mathbb{N}} \mathcal{W}_j = \mathcal{W}_{j^*}$$

Wong sequences and the QWF



Theorem (QWF [Berger, Ilchmann & T. 2012])

The following statements are equivalent for square $E, A \in \mathbb{R}^{n \times n}$:

- (i) (E, A) is regular
- (ii) $\mathcal{V}^* \oplus \mathcal{W}^* = \mathbb{R}^n$
- (iii) $E\mathcal{V}^* \oplus A\mathcal{W}^* = \mathbb{R}^n$

In particular, with $\text{im } V = \mathcal{V}^*$, $\text{im } W = \mathcal{W}^*$

(E, A) regular $\Rightarrow T := [V, W]$ and $S := [EV, AW]^{-1}$ invertible

and S, T yield quasi-Weierstrass form (QWF):

$$(SET, SAT) = \left(\begin{bmatrix} I & \\ & N \end{bmatrix}, \begin{bmatrix} J & \\ & I \end{bmatrix} \right), \quad N \text{ nilpotent}$$

Calculation of Wong sequences



Remark

Wong sequences can easily be calculated with Matlab even when the matrices still contain symbolic entries (like “R”, “L”, “C”).

```
function V=getPreImage(A,S)
% returns a basis of the preimage of A of the linear space spanned by
% the columns of S, i.e.  $\text{im } V = \{ x \mid Ax \in \text{im } S \}$ 

[m1,n1]=size(A); [m2,n2]=size(S);
if m1==m2
    H=null([A,S]);
    V=colspace(H(1:n1,:));
else
    error('Both matrices must have same number of rows');
end;
```

Consistency space



$$x(0^-) \in \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\text{imp}^-} \Leftrightarrow y \equiv 0$$

Corollary from QWF

$$\mathfrak{C}_- = \mathcal{V}_-^*$$

where \mathcal{V}_-^* is the first Wong limit of (E_-, A_-) .

The differential projector



For regular (E, A) let $(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right)$.

Definition (Differential “projector”)

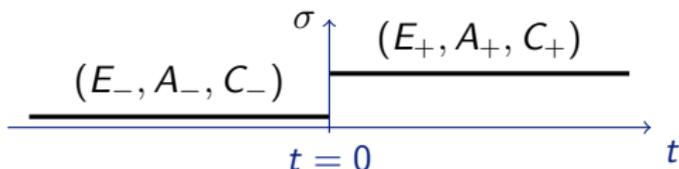
$$\Pi_{(E,A)}^{\text{diff}} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S \quad \text{and} \quad A^{\text{diff}} := \Pi_{(E,A)}^{\text{diff}} A$$

Following Implication holds:

$$x \text{ solves } E\dot{x} = Ax \quad \Rightarrow \quad \dot{x} = A^{\text{diff}}x$$

Hence, with $y = Cx$,

$$y \equiv 0 \quad \Rightarrow \quad x(0) \in \ker[C/CA^{\text{diff}}/C(A^{\text{diff}})^2/\dots/C(A^{\text{diff}})^{n-1}]$$

The spaces O_- and O_+ 

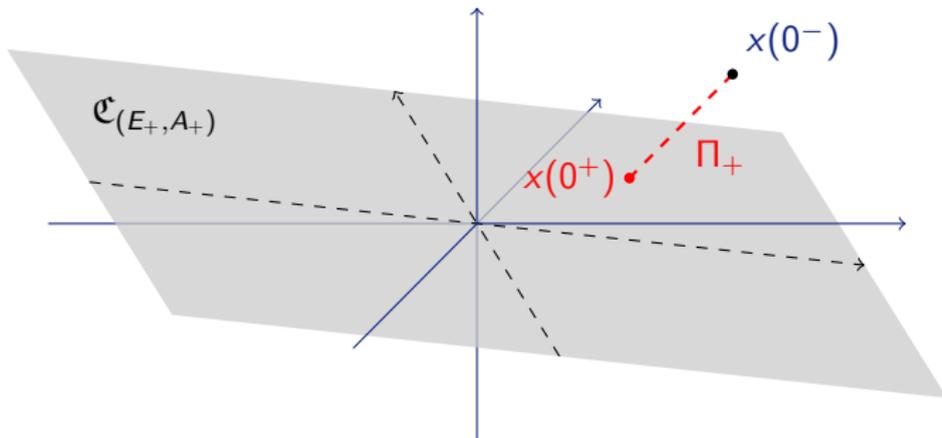
Hence

$$y_{(-\infty, 0)} \equiv 0 \quad \Rightarrow \quad x(0^-) \in \ker \underbrace{[C_- / C_- A_-^{\text{diff}} / C_- (A_-^{\text{diff}})^2 / \dots / C_- (A_-^{\text{diff}})^{n-1}]}_{:= O_-}$$

and

$$y_{(0, \infty)} \equiv 0 \quad \Rightarrow \quad x(0^+) \in \ker \underbrace{[C_+ / C_+ A_+^{\text{diff}} / C_+ (A_+^{\text{diff}})^2 / \dots / C_+ (A_+^{\text{diff}})^{n-1}]}_{:= O_+}$$

Question: $x(0^+) \in \ker O_+ \quad \Rightarrow \quad x(0^-) \in ?$

Consistency projector and O_+^- 

Assume $(S_+ E_+ T_+, S_+ A_+ T_+) = \left(\begin{bmatrix} I & 0 \\ 0 & N_+ \end{bmatrix}, \begin{bmatrix} J_+ & 0 \\ 0 & I \end{bmatrix} \right)$:

Consistency projector

$x(0^+) = \Pi_+ x(0^-)$ where

$$\Pi_+ := T_+ \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_+^{-1}$$

$$x(0^+) \in \ker O_+$$

$$\Rightarrow x(0^-) \in \Pi_+^{-1} \ker O_+ = \ker \underbrace{O_+ \Pi_+}_{=: O_+^-}$$

The impulsive effect



Assume $(S_+ E_+ T_+, S_+ A_+ T_+) = \left(\begin{bmatrix} I & 0 \\ 0 & N_+ \end{bmatrix}, \begin{bmatrix} J_+ & 0 \\ 0 & I \end{bmatrix} \right)$:

Definition (Impulse “projector”)

$$\Pi_+^{\text{imp}} := T_+ \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S_+ \quad \text{and} \quad \boxed{E_+^{\text{imp}} := \Pi_+^{\text{imp}} E_+}$$

Impulsive part of solution:

$$x[0] = - \sum_{i=0}^{n-1} (E_+^{\text{imp}})^{i+1} x(0^-) \delta_0^{(i)}$$

← Dirac impulses

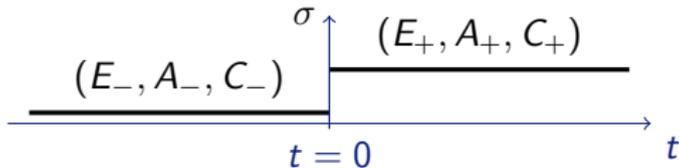
Conclusion:

$$y[0] = 0 \quad \Rightarrow \quad C_+ x[0] = 0 \quad \Rightarrow \quad \boxed{x(0^-) \in \ker O_+^{\text{imp}}}$$

where

$$O_+^{\text{imp}} := [C_+ E_+^{\text{imp}} / C_+ (E_+^{\text{imp}})^2 / \cdots / C_+ (E_+^{\text{imp}})^{n-1}]$$

Observability summary



$$y \equiv 0 \quad \Leftrightarrow \quad x(0^-) \in \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\text{imp}-}$$

with

- $\mathfrak{C}_- = \mathcal{V}_-^*$ (first Wong limit)
- $O_- = [C_- / C_- A_-^{\text{diff}} / C_- (A_-^{\text{diff}})^2 / \dots / C_- (A_-^{\text{diff}})^{n-1}]$
- $O_+^- = [C_+ / C_+ A_+^{\text{diff}} / C_+ (A_+^{\text{diff}})^2 / \dots / C_+ (A_+^{\text{diff}})^{n-1}] \Pi_+$
- $O_+^{\text{imp}} = [C_+ E_+^{\text{imp}} / C_+ (E_+^{\text{imp}})^2 / \dots / C_+ (E_+^{\text{imp}})^{n-1}]$

Example revisited



System 1:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x$$

$$y = [0 \quad 0 \quad 1] x$$

$\sigma(\cdot) : 1 \rightarrow 2$ gives

$$\mathfrak{C}_- = \text{span}\{e_1, e_3\},$$

$$\ker O_- = \text{span}\{e_1, e_2\}$$

$$\ker O_+^- = \text{span}\{e_1, e_2, e_3\},$$

$$\ker O_+^{\text{imp}} = \text{span}\{e_2, e_3\}$$

$$\Rightarrow \mathcal{M} = \{0\}$$

System 2:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x$$

$$y = [0 \quad 0 \quad 1] x$$

$\sigma(\cdot) : 2 \rightarrow 1$ gives

$$\mathfrak{C}_- = \text{span}\{e_2\},$$

$$\ker O_- = \text{span}\{e_1, e_2\}$$

$$\ker O_+^- = \text{span}\{e_1, e_2\},$$

$$\ker O_+^{\text{imp}} = \text{span}\{e_1, e_2, e_3\}$$

$$\Rightarrow \mathcal{M} = \text{span}\{e_2\}$$

Overall summary



$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned} \quad (\text{swDAE})$$

Piecewise-smooth distributional solution framework

$$x \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}^n, \quad u \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}^m, \quad y \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}^p$$

- Existence and uniqueness of solutions? ✓
- Jumps and impulses in solutions? ✓
- Conditions for impulse free solutions? ✓
- Control theoretical questions
 - Stability ✓ and stabilization
 - Observability ✓ and observer design ✓
 - Controllability ✓ and controller design

Major future challenge

Extension to nonlinear case.