The Bang Bang Funnel Controller

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Technische Universität Kaiserslautern

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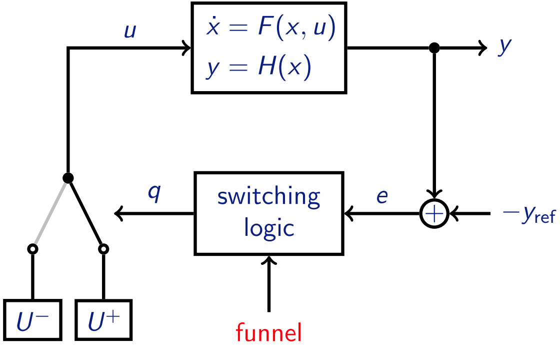
Tracking control: Closed Loop



reference signal $y_{\text{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ sufficiently smooth

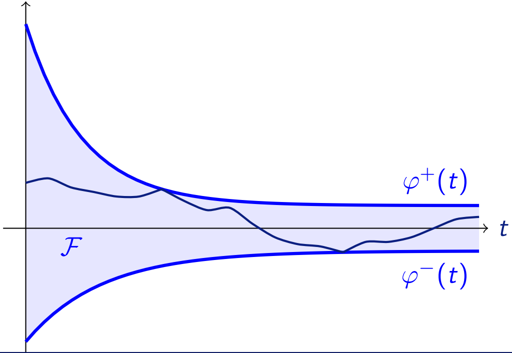
The funnel

Control goal

Error $e := y - y_{\text{ref}}$ remains within *funnel*

$$\mathcal{F} = \mathcal{F}(\varphi^-, \varphi^+) := \{ \ (t, e) \ | \ \varphi^-(t) \leq e \leq \varphi^+(t) \ \}$$

where $\varphi_\pm : \mathbb{R}_{\geq 0} \to \mathbb{R}$ sufficiently smooth



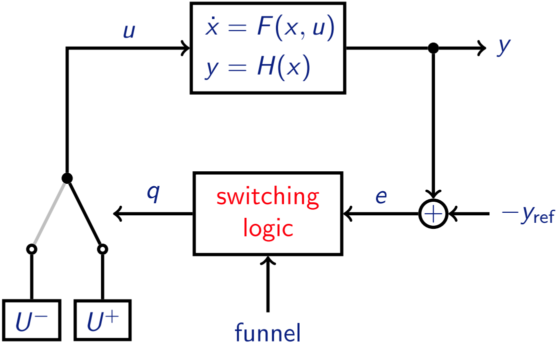
- time-varying error bound
- transient behavior
- practical convergence ($|e(t)| < \lambda$ for $t >> 0$)

The bang-bang funnel controller

Continuous funnel controller: Introduced by Ilchmann et al. 2002

New approach

Achieve Control objectives with bang-bang control, i.e. $u(t) \in \{U^-, U^+\}$



Relative degree one

Definition (relative degree one)

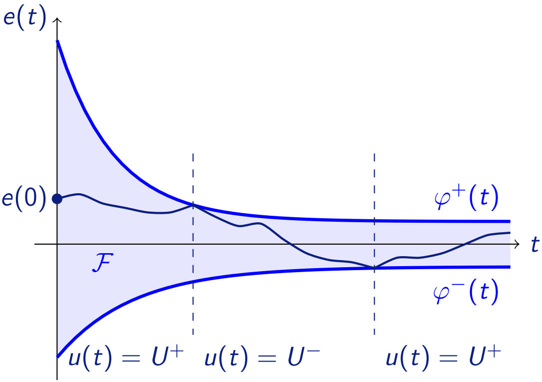
$$\begin{aligned} \dot{x} &= F(x,u) \\ y &= H(x) \end{aligned} \qquad \cong \qquad \begin{aligned} \dot{y} &= f(y,z) + \overbrace{g(y,z)}^{>0} u \\ \dot{z} &= h(y,z) \end{aligned}$$

- structural assumption
- f, g, h unknown to controller
- feasibility assumptions (later) formulated in terms of f, g, h

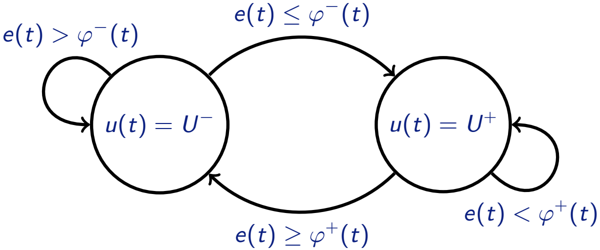
Crucial property

$$\begin{aligned} u(t) << 0 \quad &\Rightarrow \quad \dot{y}(t) << 0 \\ u(t) >> 0 \quad &\Rightarrow \quad \dot{y}(t) >> 0 \end{aligned}$$

Switching logic: illustration



Switching logic: formal definition



Too simple?

$\Rightarrow$ feasibility assumptions

Feasibility assumptions

$$\dot{y} = f(y,z) + g(y,z)u, \qquad y_0 \in \mathbb{R}$$
$$\dot{z} = h(y,z), \qquad z_0 \in Z_0 \subseteq \mathbb{R}^{n-1}$$

$$Z_t := \left\{ z(t) \;\middle|\; \begin{array}{l} z : [0,t] \to \mathbb{R}^{n-1} \text{ solves } \dot{z} = h(y,z) \text{ for a} \\ z^0 \in Z_0 \text{ and for a } y : [0,t] \to \mathbb{R} \\ \text{with } \varphi^-(\tau) \leq y(\tau) - y_{\text{ref}}(\tau) \leq \varphi^+(\tau) \\ \forall \tau \in [0,t] \end{array} \right\}.$$

Feasibility assumptions

$$\forall t \geq 0 \ \forall z_t \in Z_t : \quad \begin{array}{l} U^- \leq \dfrac{\dot{\varphi}^+(t) + \dot{y}_{\text{ref}}(t) - f(y_{\text{ref}}(t) + \varphi^+(t), z_t)}{g(y_{\text{ref}}(t) + \varphi^+(t), z_t)} \\ U^+ \geq \dfrac{\dot{\varphi}^-(t) + \dot{y}_{\text{ref}}(t) - f(y_{\text{ref}}(t) + \varphi^-(t), z_t)}{g(y_{\text{ref}}(t) + \varphi^-(t), z_t)} \end{array}$$

Main result for relative-degree-one case

Theorem (Bang-bang funnel controller, Liberzon & T. 2010)

relative degree one & funnel & simple switching logic & feasibility

$\Rightarrow$

bang-bang funnel controller works:

- *existence and uniqueness of a global solution*
- *error remains within the funnel*
- *no zeno behavior*

Necessary system knowledge

- for controller implementation
  - relative degree one
  - signals: error $e(t)$ and funnel boundaries $\varphi^{\pm}(t)$
- to check feasibility assumptions:
  - bounds for zero dynamics
  - bounds for f and g
  - bounds for y_{ref} and $\dot{y}_{\text{ref}}$
  - bounds for funnel boundaries

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Relative degree two

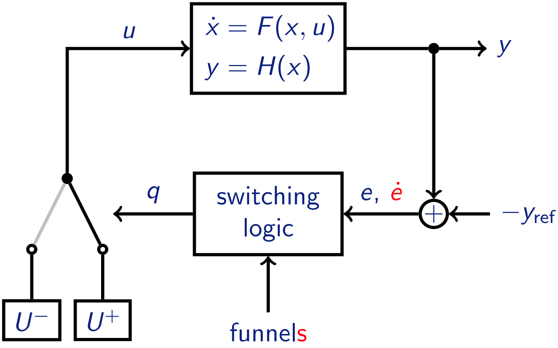
Definition (relative degree two)

$$\begin{aligned} \dot{x} &= F(x,u) \\ y &= H(x) \end{aligned} \quad \cong \quad \begin{aligned} \ddot{y} &= f(y,\dot{y},z) + \overbrace{g(y,\dot{y},z)}^{>0} u \\ \dot{z} &= h(y,\dot{y},z) \end{aligned}$$

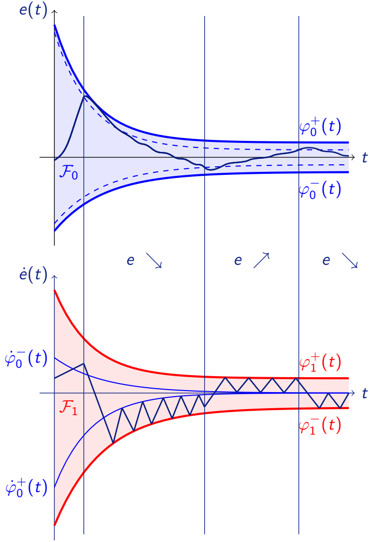
Crucial property

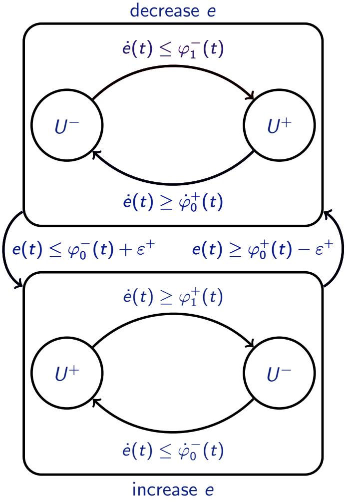
$$\begin{aligned} u(t) << 0 \quad &\Rightarrow \quad \ddot{y}(t) << 0 \\ u(t) >> 0 \quad &\Rightarrow \quad \ddot{y}(t) >> 0 \end{aligned}$$

Tracking control: Closed loop



Switching logic





Relative degree r

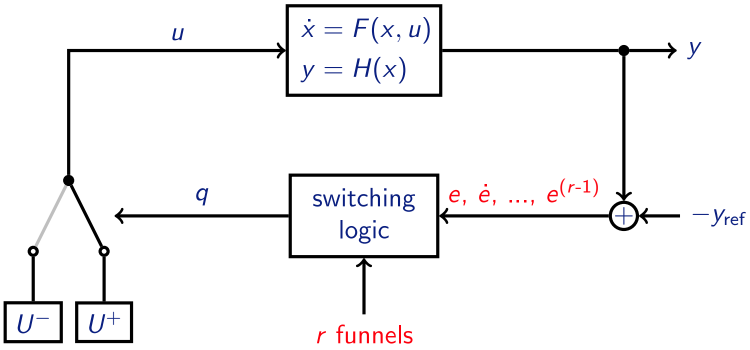
Definition (relative degree r)

$$\begin{aligned}\dot{x} &= F(x,u)\\ y &= H(x)\end{aligned} \quad\cong\quad \begin{aligned} y^{(r)} &= f(y,\dot{y},\ldots,y^{(r-1)},z) + \overbrace{g(y,\ldots,y^{(r-1)},z)}^{>0}\,u\\ \dot{z} &= h(y,\dot{y},\ldots,y^{(r-1)},z)\end{aligned}$$

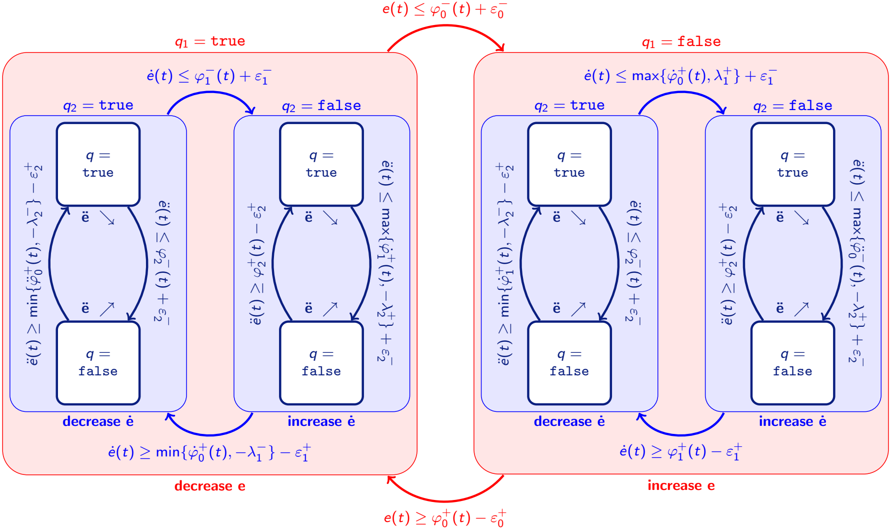
Crucial property

$$u(t) << 0 \quad\Rightarrow\quad y^{(r)}(t) << 0$$
$$u(t) >> 0 \quad\Rightarrow\quad y^{(r)}(t) >> 0$$

Tracking control: Closed loop



Recursive Approach, Example $r = 3$



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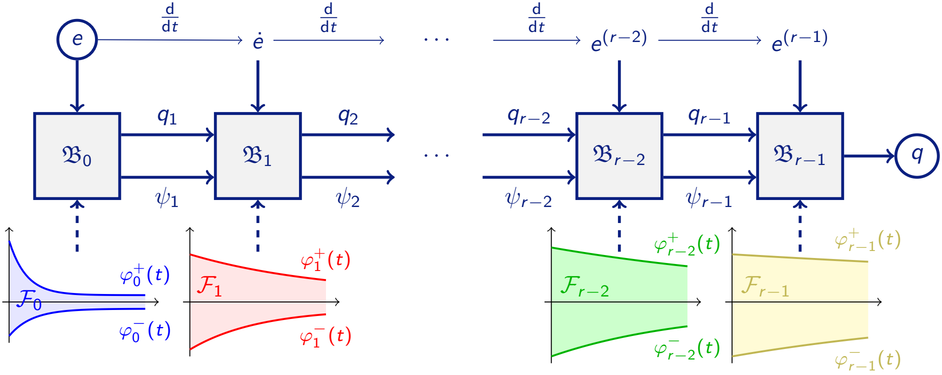
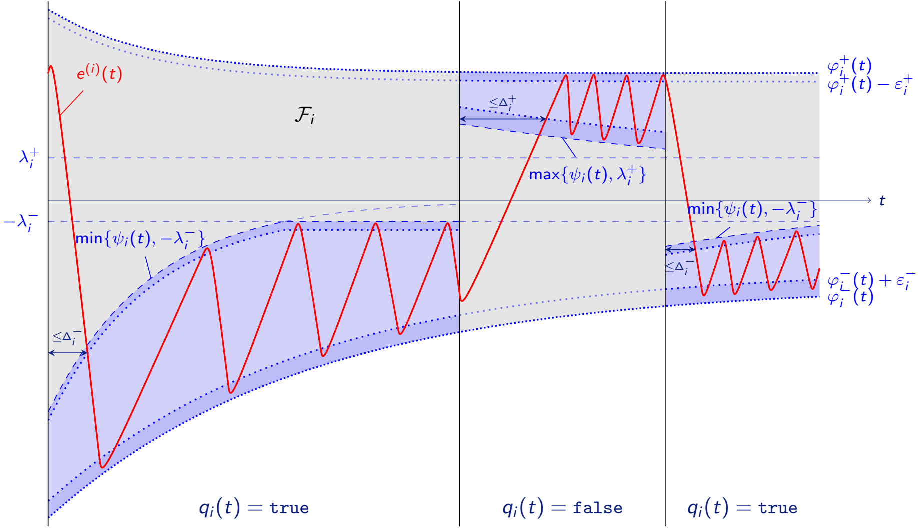
Hierarchical switching logic



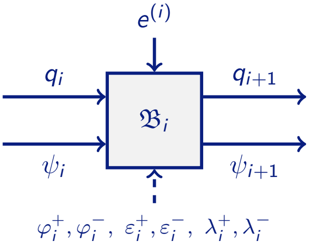
$$q_i = \texttt{true} \quad \Rightarrow \quad \text{Goal: } e^{(i)}(t) < \min\{\psi_i(t), -\lambda_i^-\}$$

$$q_i = \texttt{false} \quad \Rightarrow \quad \text{Goal: } e^{(i)}(t) > \max\{\psi_i(t), \lambda_i^+\}$$

Desired behavior of block $\mathfrak{B}_i$



Definition of switching logic

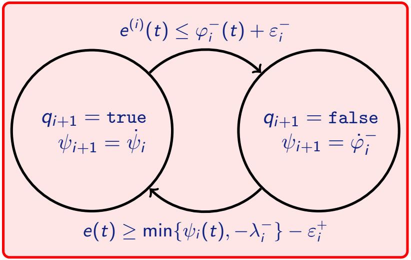


$$q_i = \texttt{true} \quad \Rightarrow \quad \text{Goal: } e^{(i)} < \min\{\psi_i, -\lambda_i^-\}$$

$$q_i = \texttt{false} \quad \Rightarrow \quad \text{Goal: } e^{(i)} > \max\{\psi_i, \lambda_i^+\}$$

where $\psi_i \in \left\{\dot\varphi_{i-1}^\pm, \ddot\varphi_{i-2}^\pm, \ldots, (\varphi_0^\pm)^{(i)}\right\}$

$q_i = \texttt{true}$



$q_i = \texttt{false}$

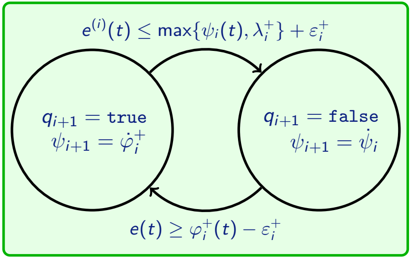
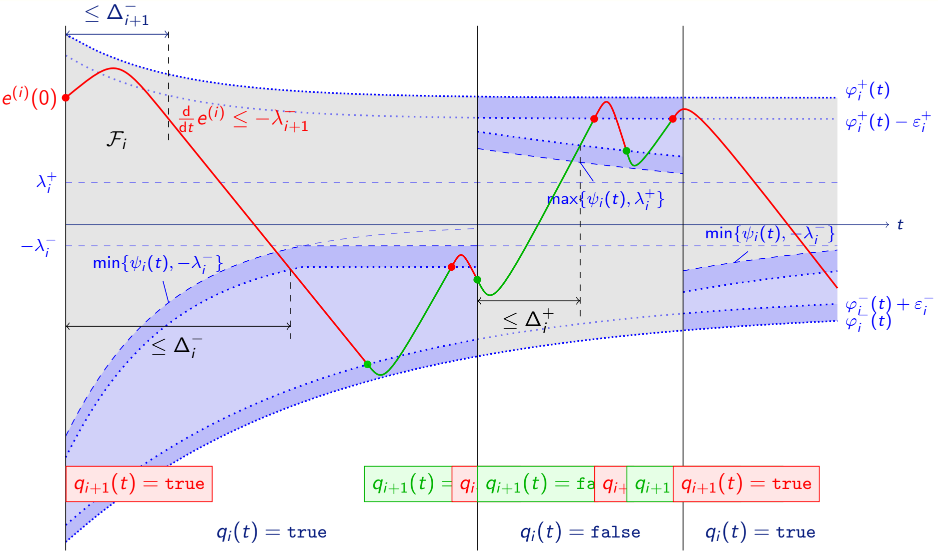


Illustration of switching logic of block $\mathfrak{B}_i$



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Feasibility assumptions **(F1)-(F8)**

„Ingredients“ for bang-bang funnel controller

- Existence and knowledge of relative degree
- Error e and its derivatives (up to $(r-1)$-st)
- Funnel boundaries $\varphi_i^\pm$ and its derivatives
- Safety distances $\varepsilon_i^\pm$ and minimal change rates $\lambda_i^\pm$
- For analysis: Settling times $\Delta_i^\pm$

Feasibility assumption (F1): Relative degree r

$$\begin{aligned} \dot{x} &= F(x,u) \\ y &= H(x) \end{aligned} \quad \cong \quad \begin{aligned} y^{(r)} &= f(y,\dot{y},\ldots,y^{(r-1)},z) + \overbrace{g(y,\ldots,y^{(r-1)},z)}^{>0}\, u \\ \dot{z} &= h(y,\dot{y},\ldots,y^{(r-1)},z), \quad z_0 \in Z_0 \subseteq \mathbb{R}^{n-r} \end{aligned}$$

and no finite escape of zero dynamics

Feasibility assumptions **(F1)-(F8)**

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Feasibility assumption (F2): reference signal

y_{ref} is r times (weakly) differentiable

Because of relative-degree assumption: y is by definition r times (weakly) differentiable, hence $e = y - y_{\text{ref}}$ is also r times (weakly) differentiable.

Feasibility assumptions **(F1)-(F8)**

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- For analysis: Settling times $\Delta_i^{\pm}$

Feasibility assumption (F3): Initial error

$$e^{(i)}(0) \in [\varphi_i^-(0) + \varepsilon_i^+, \varphi_i^-(0) - \varepsilon_i^+]$$

Feasibility assumptions **(F1)-(F8)**

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Feasibility assumption (F4): Funnel boundaries

- $\varphi_i^{\pm}$ are $r-i$ times (weakly) differentiable
- $\varphi_i^{\pm}$ and its derivates are bounded

Feasibility assumptions **(F1)-(F8)**

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- For analysis: Settling times $\Delta_i^\pm$

Feasibility assumption (F5): Funnels suitable

- $\varphi_0^+(t) - \varepsilon_0^+ > \varphi_0^-(t) + \varepsilon_0^-$
- For $\psi^\pm \in \{\dot{\varphi}_{i-1}^\pm, \ddot{\varphi}_{i-2}^\pm, \ldots, (\varphi_0^\pm)^{(i)}\}$ and $i = 1, 2, \ldots, r-1$:
  - $\varphi_i^+(t) - \varepsilon_i^+ > \max\{\psi^-(t), \lambda_i^+\} + \varepsilon_i^-$
  - $\min\{\psi^+(t), \lambda_i^-\} - \varepsilon_i^+ > \varphi_i^-(t) + \varepsilon_i^-$

Feasibility assumptions **(F1)-(F8)**

„Ingredients" for bang-bang funnel controller

- Existence and knowledge of relative degree
- Error e and its derivatives (up to $(r-1)$-st)
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- For analysis: Settling times $\Delta_i^\pm$

Feasibility assumption (F6): Settling times

For $i = 0, 1, \ldots, r-1$ and $\Delta_r^\pm \geq 0$ exists $\Delta_i^\pm$ with

$$\Delta_i^\pm \geq \Delta_{i+1}^\pm + \frac{\|\varphi_i^+\|_\infty + \|\varphi_i^-\|_\infty}{\lambda_{i+1}^\pm}$$

"restrictive" condition only together with the following assumption ...

Feasibility assumptions **(F1)-(F8)**

„Ingredients" for bang-bang funnel controller

- Existence and knowledge of relative degree
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- Funnel boundaries $\varphi_i^\pm$ and its derivatives
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- For analysis: Settling times $\Delta_i^\pm$

Feasibility assumption (F7): Safety distance

For $i = 0, 1, \ldots, r-2$ and $\psi^\pm \in \{\varphi_i^\pm, \dot\varphi_{i-1}^\pm, \ldots, (\varphi_0^\pm)^{(i)}\}$

$$\varepsilon_i^\pm > \Delta_{i+2}^\pm \|\dot\psi^\pm - \varphi_{i+1}^\pm\|_\infty + \frac{(\|\dot\psi^\pm\|_\infty + \varphi_{i+1}^\pm\|_\infty)^2}{2\lambda_{i+2}^\mp}$$

Remark: $\lambda_r^\pm$ is additional parameter for analysis and satisfies ...

Feasibility assumptions **(F1)-(F8)**

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- For analysis: Settling times $\Delta_i^\pm$

Feasibility assumption (F8): Last change rate

$$\lambda_r^+ > \max\left\{\dot{\varphi}_{r-1}^-, \ddot{\varphi}_{r-2}^-, \ldots, (\varphi_0^-)^{(r)}\right\}$$
$$-\lambda_r^- < \min\left\{\dot{\varphi}_{r-1}^+, \ddot{\varphi}_{r-2}^+, \ldots, (\varphi_0^+)^{(r)}\right\}$$

Feasibility assumptions **(F1)-(F8)**

„Ingredients“ for bang-bang funnel controller

- Existence and knowledge of relative degree
- Error e and its derivatives (up to $(r-1)$-st)
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- For analysis: Settling times $\Delta_i^\pm$

Remarks:

- Assumptions **(F1)-(F8)** independent of U^+ and U^-
- Also independent of system parameters
- For (almost) arbitrary given funnel $\mathcal{F}_0$ the remaining funnels can be constructed such that **(F3)-(F8)** are satisfied

Feasibility assumption (F9)

$$\begin{aligned} \dot{x} &= F(x,u) \\ y &= H(x) \end{aligned} \quad \cong \quad \begin{aligned} y^{(r)} &= f(y,\dot{y},\ldots,y^{(r-1)},z) + g(y,\ldots,y^{(r-1)},z)u \\ \dot{z} &= h(y,\dot{y},\ldots,y^{(r-1)},z), \quad z_0 \in Z_0 \subseteq \mathbb{R}^{n-r} \end{aligned}$$

Feasibility assumption (F9): Input "strong" enough

$$U^+ \geq \frac{\lambda_r^+ + y_{\text{ref}}^{(r)}(t) - f(y_t^0, y_t^1, \ldots, y_t^{r-1}, z_t)}{g(y_t^0, y_t^1, \ldots, y_t^{r-1}, z_t)}$$

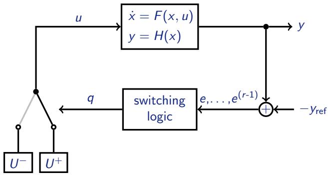
$$U^- \leq \frac{-\lambda_r^- + y_{\text{ref}}^{(r)}(t) - f(y_t^0, y_t^1, \ldots, y_t^{r-1}, z_t)}{g(y_t^0, y_t^1, \ldots, y_t^{r-1}, z_t)}$$

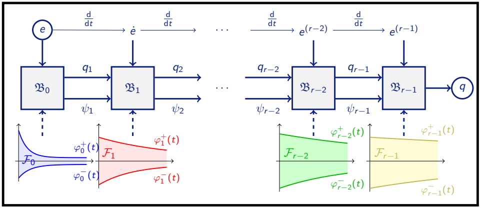
for all $t \geq 0$, $(y_t^0, y_t^1, \ldots, y_t^{r-1}) \in \Phi_t^{y_{\text{ref}}}$, $z_t \in Z_t^{y_{\text{ref}}}$

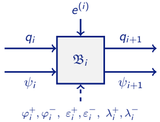
$$\Phi_t^{y_{\text{ref}}} := \left\{ (y_0,\ldots,y_{r-1}) \;\middle|\; \forall i:\; y_i - y_{\text{ref}}^{(i)}(t) \in [\varphi_i^-(t), \varphi_i^+(t)] \right\},$$

$$Z_t^{y_{\text{ref}}} := \left\{ z(t) \;\middle|\; \begin{array}{l} z \text{ solves } \dot{z} = h(y,\dot{y},\ldots,y^{(r-1)},z), z(0) = z_0 \in Z_0, \\ y \in \mathcal{C}^{r-1} \text{ with } (y(\tau),\ldots,y^{(r-1)}(\tau)) \in \Phi_\tau^{y_{\text{ref}}},\ \tau \in [0,t] \end{array} \right\}$$

Main result







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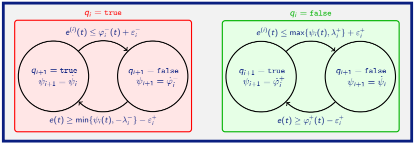

Theorem (Liberzon & T. 2013)

- *Feasibility (F1)-(F9)* $\Rightarrow$ *bang-bang funnel controller works*
- *Almost arbitrary* $\mathcal{F}_0$ + *BIBO zero dynamics* + *boundedness of* y_{ref}
  $\Rightarrow$ *Feasibility holds with sufficiently large* U^+ *and* U^-

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Simulation for $r = 4$

Example (academic), finite escape time for y possible:

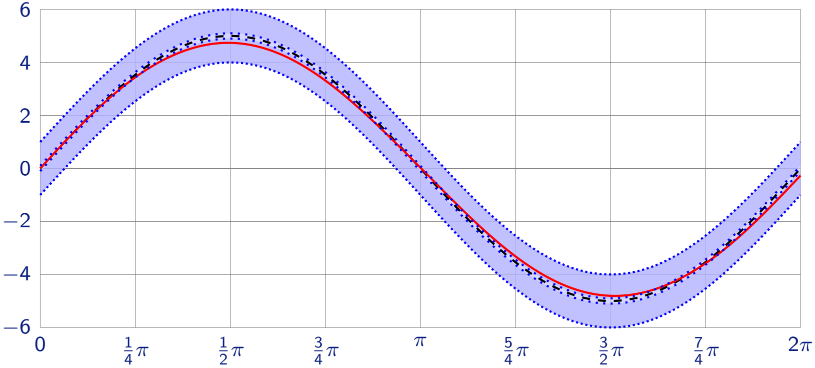
$$y^{(4)} = z\,\dddot{y}^2 + e^z u, \qquad y^{(i)}(0) = y_{\text{ref}}^{(i)}(0),\ \ i = 0,1,2,3,$$
$$\dot{z} = z(a-z)(z+b) - cy, \qquad z(0) = 0,$$
$$y_{\text{ref}}(t) = 5\sin(t)$$

controller parameter (constant funnels):

$$\begin{aligned}
&\varphi_0^+ = -\varphi_0^- \equiv 1, && \varepsilon_0^+ = \varepsilon_0^- = 0.9, && && \Delta_0^+ = \Delta_0^- = \infty,\\
&\varphi_1^+ = -\varphi_1^- \equiv 0.5, && \varepsilon_1^+ = \varepsilon_1^- = 0.1, && \lambda_1^+ = \lambda_1^- = 0, && \Delta_1^+ = \Delta_1^- = \Delta_0^\pm/2 = \infty,\\
&\varphi_2^+ = -\varphi_2^- \equiv 0.5, && \varepsilon_2^+ = \varepsilon_2^- = 0.1, && \lambda_2^+ = \lambda_2^- = 0.2, && \Delta_2^+ = \Delta_2^- = 0.4,\\
&\varphi_3^+ = -\varphi_3^- \equiv 4.5, && \varepsilon_3^+ = \varepsilon_3^- = 0.1, && \lambda_3^+ = \lambda_3^- = 4, && \Delta_3^+ = \Delta_3^- = 0.1,\\
& && && \lambda_4^+ = \lambda_4^- = 102, && \Delta_4^+ = \Delta_4^- = 0.0001.
\end{aligned}$$

$$U^+ = -U^- = 254$$

Simulation results, reference tracking



Switching frequency: up to 1000 Hz
Total number of switchings: ca. 2200

Simulation results, error plots