Controllability notions for switched DAEs

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Switched DAEs

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t)$$

or short (and more general)

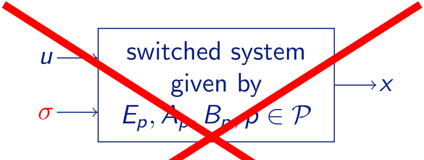
$$\boxed{E_\sigma \dot{x} = A_\sigma x + B_\sigma u} \qquad \textbf{(swDAE)}$$

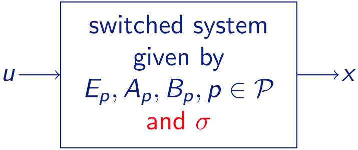
Assumptions:

- switching signal $\sigma : \mathbb{R} \to \mathcal{P}$ piecewise-constant
  in particular, no accumulation of switching times
- each matrix pair (E_p, A_p), $p \in \mathcal{P}$, is regular, i.e. $\det(sE_p - A_p) \not\equiv 0$
- piecewise-smooth distributional solution framework [T. 2009]
  i.e. $x \in \mathbb{D}^n_{\text{pw}\mathcal{C}^\infty}$, $u \in \mathbb{D}^m_{\text{pw}\mathcal{C}^\infty}$

$$\mathbb{D}_{\text{pw}\mathcal{C}^\infty} = \left\{ D = f_{\mathbb{D}} + \sum_{t \in T} D_t \;\middle|\; \begin{array}{l} f \text{ is piecewise smooth, } T \subseteq \mathbb{R} \text{ discrete} \\ \forall t \in T : D_t \in \operatorname{span}\{\delta_t, \delta_t', \delta_t'', \ldots\} \end{array} \right\}$$

Controllability for switched systems





Controllability 1

~~$\forall x_0, x_1 \; \exists (u, \sigma)$ which connects x_0, x_1~~

Controllability 2

$\forall x_0, x_1 \; \exists u$ which connects x_0, x_1 ?

Role of switching signal

Two possible viewpoints:

1. ~~σ is control input $\rightarrow$ nonlinear control problem~~
2. σ given $\rightarrow$ linear time-varying control problem

Controllability in the behavioral sense

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u \qquad \textbf{(swDAE)}$$

Definition (Distributional solution behavior)

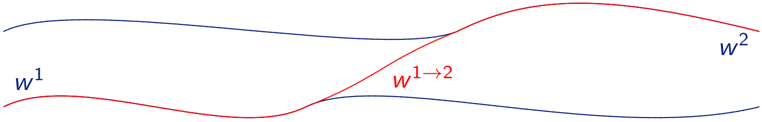
$$\mathcal{B}_\sigma := \left\{ w := (x,u) \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}^{n+m} \;\middle|\; E_\sigma \dot{x} = A_\sigma x + B_\sigma u \right\}$$

Definition (Controllability (from $t = 0$))

(swDAE) controllable $:\Leftrightarrow$ $\mathcal{B}_\sigma$ is controllable, i.e.

$$\forall w^1, w^2 \in \mathcal{B}_\sigma \; \exists T \geq 0 \; \exists w^{1\to 2} \in \mathcal{B}_\sigma :$$

$$w^{1\to 2}_{(-\infty,0)} = w^1_{(-\infty,0)} \quad \wedge \quad w^{1\to 2}_{(T,\infty)} = w^2_{(T,\infty)}$$



Comments on Controllability

Instantaneous Control

The definition allows $T = 0$ for two reasons:

1. Dirac impulse in u $\Rightarrow$ Jump in x
2. Switch & Inconsistency $\Rightarrow$ Jump in x

Lemma (Controllability to origin)

(swDAE) *controllable* $\Leftrightarrow$

$$\forall w \in \mathcal{B}_\sigma\ \exists T \geq 0\ \exists w^0 \in \mathcal{B}_\sigma : \quad w^0_{(-\infty,0)} = w_{(0,\infty)} \ \wedge\ w^0_{(T,\infty)} = 0$$

Definition (Controllability subspace)

$$\mathcal{C}_\sigma := \{\ x_0 \in \mathbb{R}^n \mid \exists (x,u) \in \mathcal{B}_\sigma\ \exists T \geq 0 : x(0-) = x_0\ \wedge x(T+) = 0\ \}$$

Consistency

(swDAE) controllable $\not\Rightarrow$ $\mathcal{C}_\sigma = \mathbb{R}^n$

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Regular DAEs and the quasi-Weierstrass form

Theorem ((Quasi-)Weierstrass form, [Weierstrass 1868])

(E, A) *is regular*
$\Leftrightarrow \exists S, T$ *invertible:* $(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right)$, *$N$ nilpotent*

Calculate S, T via Wong-sequences [Wong 1974; Berger, Ilchmann, T. 2012]

Definition (Some useful "projectors")

Consistency projector: $\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$

Differential projector: $\Pi^{\text{diff}}_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S$

Impulse projector: $\Pi^{\text{imp}}_{(E,A)} := T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S$

$A^{\text{diff}} := \Pi^{\text{diff}} A$, $B^{\text{diff}} := \Pi^{\text{diff}} B$, $E^{\text{imp}} := \Pi^{\text{imp}} E$, $B^{\text{imp}} := \Pi^{\text{imp}} B$

$\text{im} \subseteq \text{im}\,\Pi_{(E,A)} = \mathcal{V}^*$, $\text{im} \subseteq \ker \Pi_{(E,A)} = \mathcal{W}^*$

Controllability characterization

Theorem (T. 2012)

(x,u) smooth solution of (E,A) $\Leftrightarrow \exists c \in \mathbb{R}^n\ \forall t \in \mathbb{R}$:

$$x(t) = e^{A^{\text{diff}}t}\Pi_{(E,A)}c + \int_0^t e^{A^{\text{diff}}(t-s)}B^{\text{diff}}u(s)\,\mathrm{d}s - \sum_{i=0}^{n-1}(E^{\text{imp}})^i B^{\text{imp}} u^{(i)}(t)$$

Corollary

Consistency space: $\operatorname{im}\Pi_{(E,A)} \oplus \operatorname{im}\langle E^{\text{imp}}, B^{\text{imp}}\rangle$

Controllability space: $\langle A^{\text{diff}}, B^{\text{diff}}\rangle \oplus \operatorname{im}\langle E^{\text{imp}}, B^{\text{imp}}\rangle$

where $\langle A, B\rangle := [B, AB, A^2B, \ldots, A^{n-1}B]$

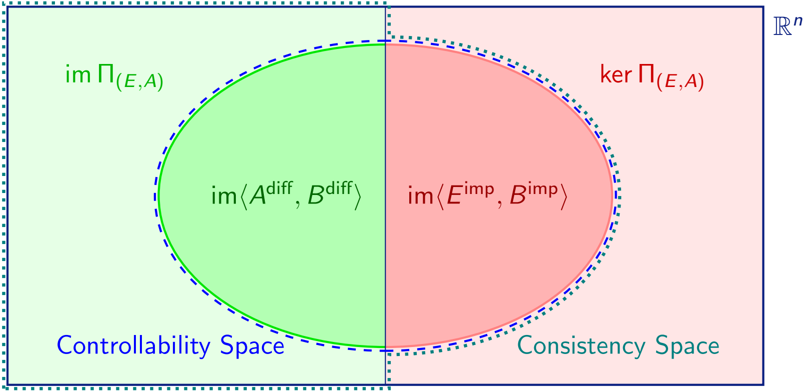
Theorem (Controllability characterization)

$E\dot{x} = Ax + Bu$ *controllable (in the behavioral sense)*

$\Leftrightarrow \quad \langle A^{\text{diff}}, B^{\text{diff}}\rangle = \operatorname{im}\Pi_{(E,A)}$

$\Leftrightarrow \quad \langle A^{\text{diff}}, B^{\text{diff}}\rangle + \ker\Pi_{(E,A)} = \mathbb{R}^n$

Overall picture



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Controllability characterization: Single switch case

$$\boxed{E_\sigma \dot{x} = A_\sigma x + B_\sigma u} \qquad \textbf{(swDAE)}$$

Consider switching signal with one switch:

$$\sigma_1^\varepsilon(t) := \begin{cases} 1, & t < \varepsilon \\ 2, & t \geq \varepsilon \end{cases}$$

Need $\varepsilon > 0$ to allow mode 1 to act on trajectory.

Theorem (Controllability characterization)

(swDAE) *with switching signal* σ_1^ε *controllable*

$$\Leftrightarrow \quad \operatorname{im}\langle A_1^{\text{diff}}, B_1^{\text{diff}}\rangle + \Pi_{(E_2,A_2)}^{-1} \operatorname{im}\langle A_2^{\text{diff}}, B_2^{\text{diff}}\rangle \supseteq \operatorname{im} \Pi_{(E_1,A_1)}$$

$$\Leftrightarrow \quad \ker \Pi_{(E_1,A_1)} + \operatorname{im}\langle A_1^{\text{diff}}, B_1^{\text{diff}}\rangle + \Pi_{(E_2,A_2)}^{-1} \operatorname{im}\langle A_2^{\text{diff}}, B_2^{\text{diff}}\rangle = \mathbb{R}^n$$

Conclusions

- Controllability of switched DAEs
  - Distributional solution theory (jumps and Dirac impulses)
  - Controllability in the behavioral sense
  - Result for single-switch case
  - Just at the beginning of research
- Open questions and further issues
  - Multiple-switch case
  - Control via switching signal
  - Controllability of Dirac impulses
  - Duality with observability