A converse Lyapunov theorem for switched DAEs

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Content

Switched DAEs

Linear switched DAE (differential algebraic equation)

(swDAE) $\boxed{E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t)}$ or short $\boxed{E_\sigma \dot{x} = A_\sigma x}$

with

- switching signal $\sigma : \mathbb{R} \to \{1, 2, \ldots, \mathrm{p}\}$
  - piecewise constant, right-continuous
  - locally finitely many jumps (no Zeno behavior)
- matrix pairs $(E_1, A_1), \ldots, (E_\mathrm{p}, A_\mathrm{p})$
  - $E_p, A_p \in \mathbb{R}^{n\times n}$, $p = 1, \ldots, \mathrm{p}$
  - (E_p, A_p) regular, i.e. $\det(E_p s - A_p) \not\equiv 0$
  - impulse-free solutions (but jumps are allowed!)

Question

$E_\sigma \dot{x} = A_\sigma x$ asymp. stable $\forall\sigma \quad \overset{?}{\Rightarrow} \quad$ common Lyapunov function

Lyapunov norms

More general approach:

Definition (Lyapunov norm)

$|||\cdot|||$ is a λ-*Lyapunov norm*, $\lambda \in \mathbb{R}$,

$:\Leftrightarrow \forall\sigma : \boxed{|||x(t)||| \leq e^{\lambda t}|||x(0-)|||}\ \forall$ solutions x of $E_\sigma \dot{x} = A_\sigma x$

In particular: $\lambda < 0 \quad \Rightarrow \quad V = |||\cdot|||$ defines Lyapunov function

New question

Find Lyapunov norm for $E_\sigma \dot{x} = A_\sigma x$ (stable or unstable)

Solution formula

Theorem (A^{diff} and $\Pi_{(E,A)}$, Tanwani & T. 2010)

Let (E,A) be regular and consider

$$E\dot{x} = Ax \quad \text{on } [0,\infty)$$

$\Rightarrow \exists$ *unique consistency projector* $\Pi_{(E,A)}$ *and unique flow matrix* A^{diff}*:*

$$\boxed{x(0) = \Pi_{(E,A)} x(0-)} \qquad \boxed{\dot{x} = A^{\text{diff}} x} \quad \text{on} \quad (0,\infty)$$

Furthermore, $A^{\text{diff}}\Pi_{(E,A)} = \Pi_{(E,A)}A^{\text{diff}}$.

Corollary (Solution formula for switched DAE)

Any solution of the switched DAE $E_\sigma \dot{x} = A_\sigma x$ *has the form*

$$x(t) = e^{A_k^{\text{diff}}(t-t_k)}\Pi_k e^{A_{k-1}^{\text{diff}}(t_k-t_{k-1})}\Pi_{k-1}\cdots e^{A_1^{\text{diff}}(t_2-t_1)}\Pi_1 e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0 x(t_0-)$$

Evolution operator

$$x(t) = \underbrace{e^{A_k^{\text{diff}}(t-t_k)}\Pi_k e^{A_{k-1}^{\text{diff}}(t_k-t_{k-1})}\Pi_{k-1}\cdots e^{A_1^{\text{diff}}(t_2-t_1)}\Pi_1 e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0}_{=:\,\Phi^\sigma(t,t_0)} x(t_0-)$$

Let $\mathcal{M} := \{\ (A_p^{\text{diff}}, \Pi_p)\ \mid \text{corresponding to } (E_p, A_p), p = 1,\ldots,\mathtt{p}\ \}$.

Definition (Set of all evolutions with fixed time span $\Delta t > 0$)

$$\mathcal{S}_{\Delta t} := \bigcup_\sigma \{\ \Phi^\sigma(t_0+\Delta t, t_0)\ \mid\ t_0 \in \mathbb{R}\ \}$$
$$= \left\{\ \prod_{i=0}^{k} e^{A_i^{\text{diff}}\tau_i}\Pi_i\ \middle|\ (A_i^{\text{diff}}, \Pi_i) \in \mathcal{M},\ \sum_{i=0}^{k}\tau_i = \Delta t,\ \tau_i > 0\ \right\}$$

Note that $\forall t_0 \in \mathbb{R}\ \forall \Delta t > 0$:

$$x \text{ solves } E_\sigma \dot{x} = A_\sigma x \quad \Leftrightarrow \quad \exists \Phi_{\Delta t} \in \mathcal{S}_{\Delta t}: \quad x(t_0+\Delta t) = \Phi_{\Delta t} x(t_0-)$$

Semi group property

Lemma (Semi group)

The set

$$\mathcal{S} := \bigcup_{\Delta t > 0} \mathcal{S}_{\Delta t}$$

is a semi group with

$$\mathcal{S}_{s+t} = \mathcal{S}_s \mathcal{S}_t := \{ \ \Phi_s \Phi_t \ | \ \Phi_s \in \mathcal{S}_s, \Phi_t \in \mathcal{S}_t \ \}$$

Need commutativity to show "$\subseteq$":

$$e^{A^{\text{diff}}\tau}\Pi = e^{A^{\text{diff}}(\tau-\tau')} e^{A^{\text{diff}}\tau'}\Pi\Pi = e^{A^{\text{diff}}(\tau-\tau')}\Pi e^{A^{\text{diff}}\tau'}\Pi$$

for any $(A^{\text{diff}}, \Pi) \in \mathcal{M}$ and $0 < \tau' < \tau$

Exponential growth bound

Definition (Exponential growth bound)

For $t > 0$ the *exponential growth bound* of $E_\sigma \dot{x} = A_\sigma x$ is

$$\lambda_t(\mathcal{S}_t) := \sup_{\Phi_t \in \mathcal{S}_t} \frac{\ln \|\Phi_t\|}{t} \in \mathbb{R} \cup \{-\infty, \infty\}$$

Definition implies for all solutions x of $E_\sigma \dot{x} = A_\sigma x$:

$$\|x(t)\| = \|\Phi_t x(0-)\| \leq \|\Phi_t\| \, \|x(0-)\| \leq e^{\lambda_t(\mathcal{S}_t)\, t} \|x(0-)\|$$

Difference to switched ODEs without jumps

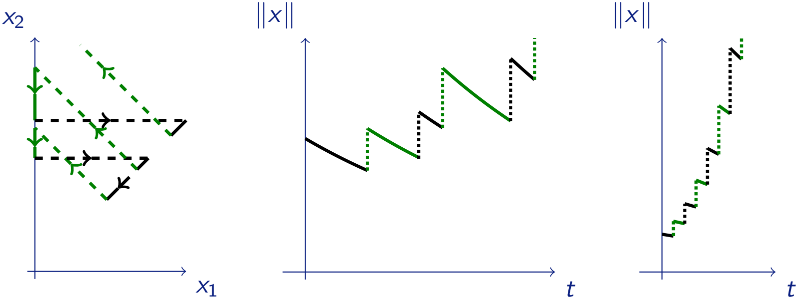
$\lambda_t(\mathcal{S}_t) = \pm\infty$ is possible!

All jumps are trivial, i.e. $\Pi_p = 0 \quad \Rightarrow \quad \lambda_t(\mathcal{S}_t) = -\infty$

Infinite exponential growth bound

Example:

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \right) \quad (E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$



For small dwell times: $\Phi_t \approx (\Pi_1 \Pi_2)^k = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^k = 2^{k-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Existence of exponential growth rate

Theorem (Boundedness of $\mathcal{S}_t$)

$\mathcal{S}_t$ is bounded $\Leftrightarrow$ the set of consistency projectors is product bounded

Reminder:

$$\mathcal{S}_t := \left\{ \prod_{i=0}^{k} e^{A_i^{\text{diff}}\tau_i}\Pi_i \;\middle|\; (A_i^{\text{diff}}, \Pi_i) \in \mathcal{M},\ \sum_{i=0}^{k} \tau_i = \Delta t,\ \tau_i > 0 \right\}$$

Theorem (Exponential growth rate well defined)

Let the consistency projectors be product bounded and not all be trivial, then the (upper) Lyapunov exponent

$$\lambda(\mathcal{S}) := \lim_{t\to\infty} \lambda_t(\mathcal{S}_t) = \lim_{t\to\infty} \sup_{\Phi_t \in \mathcal{S}_t} \frac{\|\Phi_t\|}{t}$$

of $E_\sigma \dot{x} = A_\sigma x$ is well defined and finite.

Barabanov norm

Definition (Barabanov norm)

$|||\cdot|||$ is called Barabanov norm for $E_\sigma \dot{x} = A_\sigma x$, iff

1. $|||x(t)||| = |||\Phi_t x(0-)||| \leq e^{\lambda t}|||x(0-)|||, \quad \Phi_t \in \mathcal{S}_t$
2. $\forall x^0 \in \mathbb{R}^n\ \exists \overline{\Phi}_t \in \overline{\mathcal{S}_t} : \quad |||\overline{\Phi}_t x_0||| = e^{\lambda t}|||x_0|||$

In particular, ever Barabanov norm is also a λ-Lyapunov norm, hence if $\lambda < 0$ we have a Lyapunov function

Theorem (Existence of Barabanov norm)

Assume $\mathcal{S}$ is irreducible, i.e. $\mathcal{S}\mathcal{M} \subseteq \mathcal{M}$ implies $\mathcal{M} = \emptyset$ or $\mathcal{M} = \mathbb{R}^n$. Then the following are equivalent:

1. *The consistency projectors are product bounded*
2. *The Lyapunov exponent $\lambda(\mathcal{S})$ is bounded*
3. *There exists a Barabanov norm with $\lambda = \lambda(\mathcal{S})$*

Construction of Barabanov norm

Construction of Barabanov norm similar as in (Wirth 2002, LAA):

$$S_\infty := \bigcap_{T\geq 0} \overline{\bigcup_{t\geq T} e^{-\lambda(S)\,t}\mathcal{S}_t}$$

is a compact nontrivial semigroup, the limit semigroup.

$$|||x||| := \max\left\{\ \|Sx\|\ \mid S\in\mathcal{S}_\infty\ \right\}$$

is the sought Barabanov norm.

The reducible case

Theorem (Lyapunov norm)

For each $\varepsilon > 0$

$$|||x|||_\varepsilon := \sup_{t>0} \sup_{\Phi_t \in \mathcal{S}_t} e^{-(\lambda(\mathcal{S})+\varepsilon)t} \|\Phi_t x\|$$

defines a Lyapunov norm for $E_\sigma \dot{x} = A_\sigma x$.

Corollary (Converse Lyapunov Theorem)

$E_\sigma \dot{x} = A_\sigma x$ *is uniformly exp. stable* $\Rightarrow V = |||\cdot|||_\varepsilon$ *is Lyapunov function*

In particular: $V(\Pi x) \leq V(x)$ for all consistency projectors Π

Non-smooth Lyapunov function

$|||\cdot|||_\varepsilon$ in general non-smooth. Smoothification as in Yin, Sontag & Wang 1996 might violate jump condition!

Conclusions

- Studied switched DAEs $E_\sigma \dot{x} = A_\sigma x$
- Key observation:

$$x(t) = e^{A_k^{\text{diff}}(t-t_k)}\Pi_k \cdots e^{A_1^{\text{diff}}(t_2-t_1)}\Pi_1 e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0 x(t_0-)$$

- Flow set

$$\mathcal{S}_t := \left\{ \prod_{i=0}^{k} e^{A_i^{\text{diff}}\tau_i}\Pi_i \;\middle|\; (A_i^{\text{diff}}, \Pi_i) \in \mathcal{M},\ \sum_{i=0}^{k} \tau_i = \Delta t,\ \tau_i > 0 \right\}$$

- Product boudedness of consistency projectors necessary and sufficient for boundedness of $\mathcal{S}_t$
- Construction of Barabanov norm in irreducible case
- Construction of Lyapunov norm in reducible case